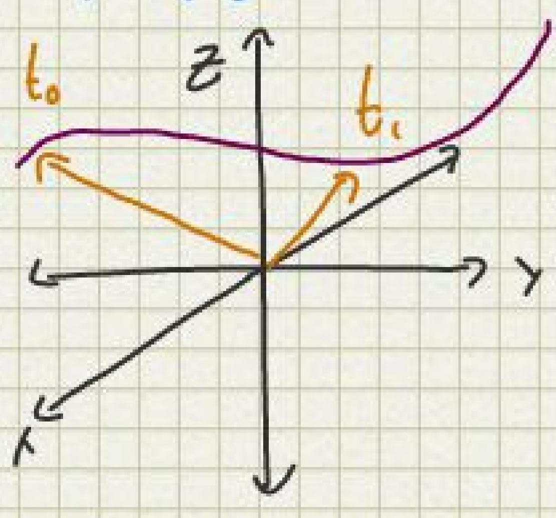
4/10/2014



$\vec{R}(t) \quad t_0 \le t \le t_1$

$$\int_c f(x,y,z) = \int_{t_0}^{t_1} f(\vec{R},t) \underbrace{\sqrt{x^2+y^2+z^2}\, dt}_{\|R'(t)\|\, dt}$$

$\int_c \vec{F} \cdot d\vec{R}$

$$\int_c \vec{F}(x,y,z)\, d\vec{R} = \int_c \vec{F} \cdot \vec{T}\, ds$$

$\vec{F} = (y-z^2)\hat{\imath} + 2yz\hat{\jmath} + x^2\hat{k}$

$$= \int_{t_0}^{t_1} \vec{F}(\vec{R}(t)) \cdot \frac{\vec{R}'(t)}{\cancel{\|\vec{R}'(t)\|}} \cancel{\|\vec{R}'(t)\|}\, dt$$

$C: \begin{cases} x = t^2 \\ y = 2t \\ z = t \end{cases}$

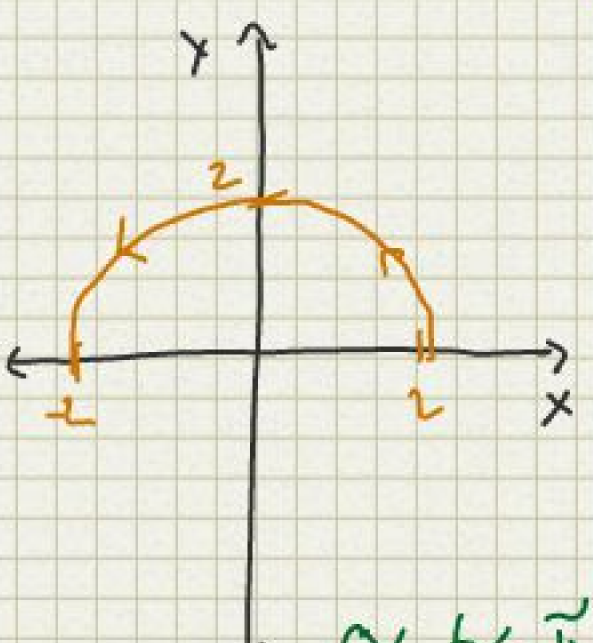
$$= \int_{t_0}^{t_1} \underbrace{\vec{F}(\vec{R}(t) \cdot \vec{R}'(t))}_{g(t)} dt$$

$0 \le t \le 1$

$$= \int_0^1 \langle (2t)^2 - t^2, 2(2t)(t) - (t^2)^3 \rangle \cdot \langle 2t, 2, 1 \rangle\, dt$$

$\vec{R}(t) = \langle 2t, 2, 1 \rangle$

$$\text{L.I.} = \int_0^1 (6t^3 + 8t^2 - t^4)\, dt = \left(\frac{6}{4}t^4 + \frac{8}{3}t^3 - \frac{1}{5}t^5\right)\Big|_0^1$$



$F = y\hat{\imath} + x\hat{\jmath}$

$C = x^2 + y^2 = 4$ from $(2,0) \to (-2,0)$

$$\int_c \vec{F}\, d\vec{R} = \int_0^{\pi} \langle 2\sin(t), 2\cos(t) \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle\, dt$$

$$= \int_0^{\pi} (-4\sin^2(t) + 4\cos^2(t))\, dt$$

$0 \le t \le \pi$

$$= \int_0^{\pi} \cos(2t)\, dt = \frac{4\sin(2t)}{2}\Big|_0^{\pi} = 0$$

$\vec{R}(t) = \begin{cases} x = 2\cos(t) \\ y = 2\sin(t) \end{cases}$

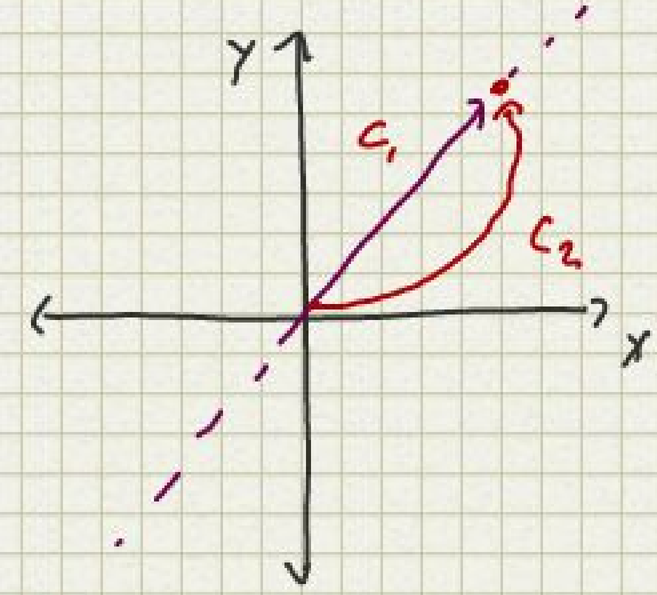
$\begin{cases} x = t \\ y = \sqrt{4-t^2} \end{cases}$

$\vec{R}'(t) = \langle -2\sin(t), 2\cos(t) \rangle$

$\int_{C_i} \vec{F}\, d\vec{R}$ $\quad F = xy^2\hat{i} + x^2y\hat{j}$

C_1 = line segment $(0,0) \to (2,4)$

C_2 = parabola $(0,0) \longrightarrow (2,4)$



$C_1 \begin{cases} x = t & 0 \le t \le 2 \\ y = 2t & R'(t) = \langle 1, 2 \rangle \end{cases}$

C_1: $\int_0^2 (\langle t(2t)^2, t^2(2t) \rangle \cdot \langle 1, 2 \rangle)\,dt$

$= \int_0^2 (4t^3 + 4t^3)\,dt = 2t^4 \Big|_0^2 = 2(16) = 32$

$C_2 \begin{cases} x = t & 0 \le t \le 2 \\ y = t^2 & R'(t) = \langle 1, 2t \rangle \end{cases}$

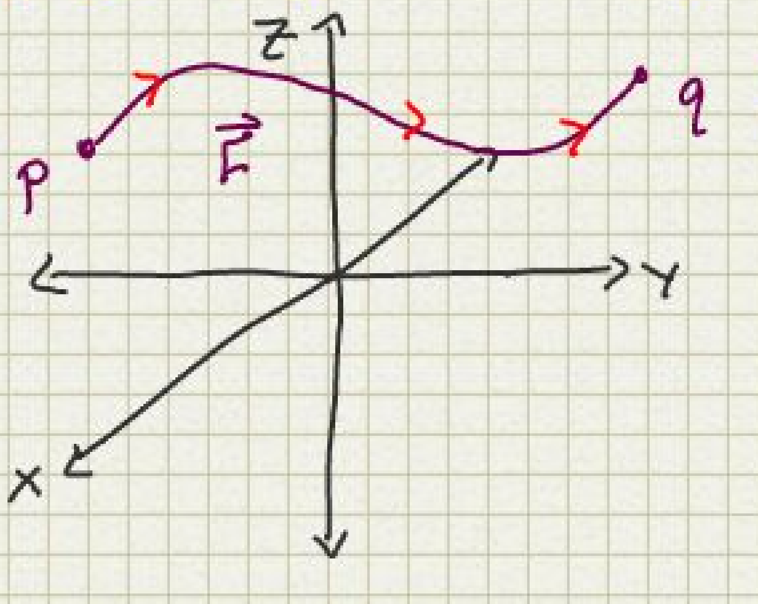
C_2: $\int_0^2 \langle t(t^2)^2, t^2t^2 \rangle \cdot \langle 1, 2t \rangle\, dt$

$= \int_0^2 (t^5 + 2t^5)\,dt = 3\int_0^2 t^5\,dt = 3\,\frac{t^6}{6}\Big|_0^2$

$= \frac{1}{2}(2)^6 = 32$

Conservative Vector Field

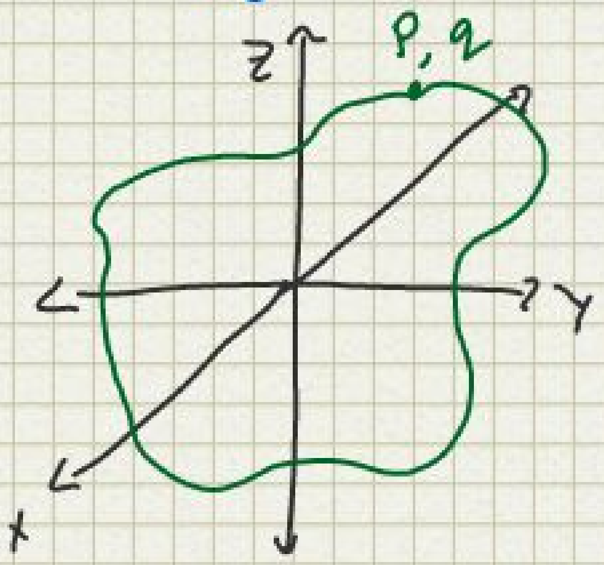
The vector is conservative if there exists a scalar function $g(x,y,z)$ such that $\vec{F} = \nabla g$. We call g the scalar potential of $\vec{F}$.



① If F is conservative then LI $\int_C \vec{F}\, d\vec{R}$ is an independent path.

$\vec{F}\, d\vec{R} \longrightarrow g(q) - g(p)$

2) $\oint_c \vec{F} \cdot d\vec{R} = 0$ (intagral of closed cycle)



$\oint_c \vec{F} \cdot d\vec{R} = g(q) - g(p) = 0$

$$\int_c \vec{F} \cdot d\vec{R} = \int_{t_0}^{t_1} F(\vec{R}(t)) \cdot \langle x', y', z' \rangle dt$$

$$= \int_{t_0}^{t_1} \langle \frac{\partial}{\partial x} g, \frac{\partial}{\partial y} g, \frac{\partial}{\partial z} g \rangle \circ \langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial t} \rangle dt$$

$$= \int_{t_0}^{t_1} \frac{dg}{\partial t} dt = g(t_1) - g(t_0) = g(q) - g(p)$$

$g = \frac{x^2 y^2}{2}$

$\nabla g = \langle xy^2, x^2 y \rangle$

$LI = \int_c \vec{F} d\vec{R} = g(2,4) - g(0,0) = 32$ (same as problem on page before)

Conservative Field Test

$g_{xy} = g_{yx}$ if $\vec{F} = \nabla g$

2D

$f_1 = g_x \quad f_{1y} = g_{xy} = 2xy$

$f_2 = g_y \quad f_{2x} = g_{yx} = 2xy$ – should be the same

$f_1 = g_x = xy^2 \quad f_2 = g_y = yx^2$

$g_x \, dx = \int xy^2 dx = \frac{1}{2} x^2 y^2 + h(y) \quad h(y) \approx 0$

$g_y \, dy = \int yx^2 dy = \frac{1}{2} x^2 y^2 + l(x) \quad l(x) \approx 0$

$g = \frac{x^2 y^2}{2} + C$

$C \approx 0$

Verify $\vec{F} = (e^x\sin(y)-y)\hat{\imath} + (e^x\cos(y)-x-2)\hat{\jmath}$

Find g : $\nabla g = F$ $\quad f_{1y} = f_{2x}$

Derivatives: $(e^x\cos(y)-1) = (e^x\cos(y)-1)$

$g = \int g_x\,dx = \int (e^x\sin(y)-y)\,dx = e^x\sin(y) - xy$ $\begin{pmatrix}\text{two elements}\\ + h(y)\end{pmatrix}$

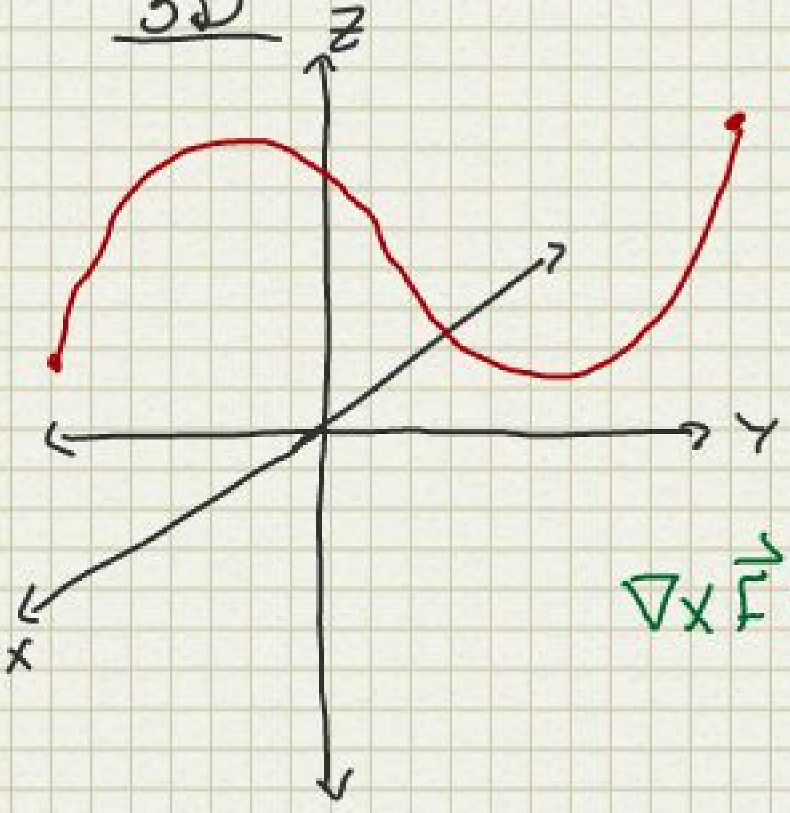
$\cup$

$g = \int g_y\,dy = \int (e^x\cos(y)-x-2)\,dy = e^x\sin(y) - xy - 2y$ $\begin{pmatrix}\text{three}\\ \text{elements}\\ + h(x)\end{pmatrix}$

g is the union between both integrals

$g = e^x\sin(y) - xy - 2y$

3D



$\vec{F} = \nabla g$

$g_x = f_1 \begin{cases} g_{xy} \\ g_{xz} \end{cases}$ $\quad f_{1y} = f_{2x}$

$g_y = f_2 \begin{cases} g_{yx} \\ g_{yz} \end{cases}$ $\quad f_{2z} = f_{3y}$

$g_z = f_3 \begin{cases} g_{zx} \\ g_{zy} \end{cases}$ $\quad f_{3x} = f_{1z}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0 \quad \text{curl} = 0$$

$$= \hat{\imath}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$g = \int f_1\,dx$

$g = \int f_2\,dy$

$g = \int f_3\,dz$

Verify Conservatives

$\vec{F} = \langle 20x^3z + 2y^2, 4xy, 5x^4 + 3z^2 \rangle \quad \nabla \times \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 20x^3z + 2y^2 & 4xy & 5x^4 + 3z^2 \end{vmatrix} = (0-0)\hat{i} - (20x^3 - 20x^3)\hat{j} + \hat{k}(4y - 4y) = \vec{0}$$

$g_x = f_1 \quad g = \int (20x^3z + 2y^2)\,dx = 5x^4z + 2y^2x + \ldots$

$g_y = f_2 \quad g = \int 4xy\,dy = 2xy^2 + \ldots$

$g_z = f_3 \quad g = \int (5x^4 + 3z^2)\,dz = 5x^4z + z^3 + \ldots$

$g = 5x^4z + 2y^2x + z^3$