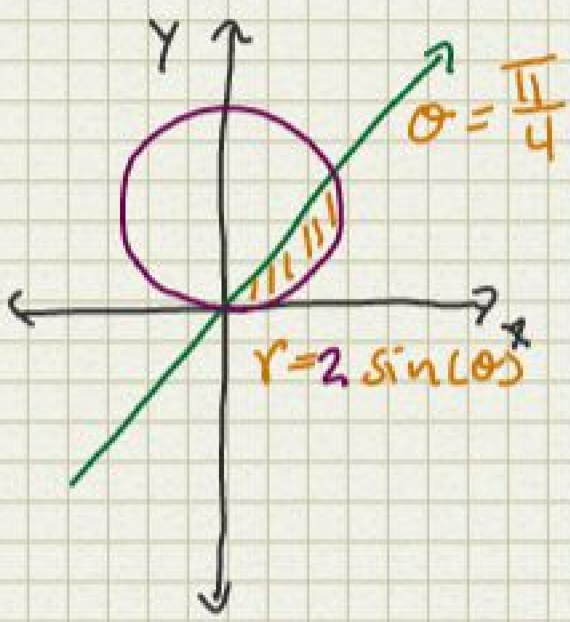
3/13/2014

$$\iint_{D(x,y)} f(x,y)\,dA = \iint_{D(\theta,R)} f(r\cos(\theta), r\sin(\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta_1}^{\theta_2} \int_{R_1(\theta}^{R_2(\theta)} f(r,\theta)\; r\,dr\,d\theta$$

compute the area below $y=x$ and above $x^2+y^2-2y=0$



$x^2+y^2-2y=0$

$x^2+(y^2-2y+1)=1$

$x^2+(y-1)^2=1$

$* \sin^2(\theta)=\frac{1}{2}(1-\cos(2\theta))$

$\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$

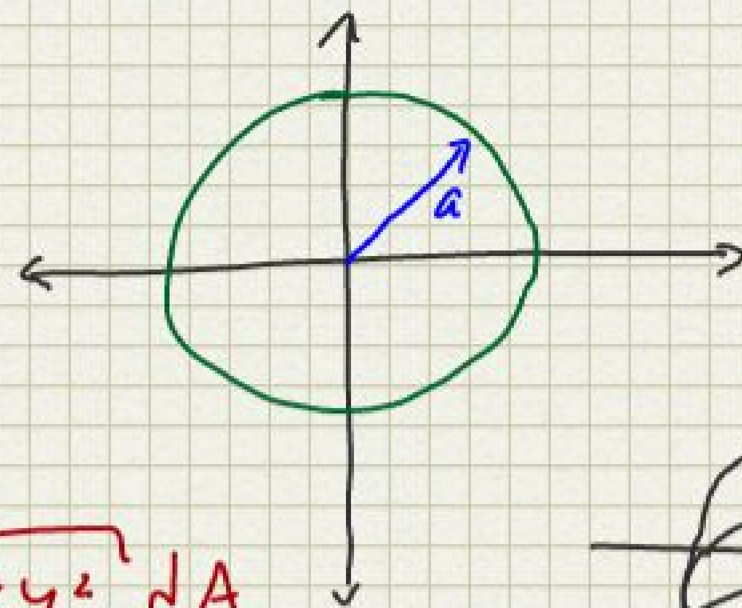
Line at x-axis

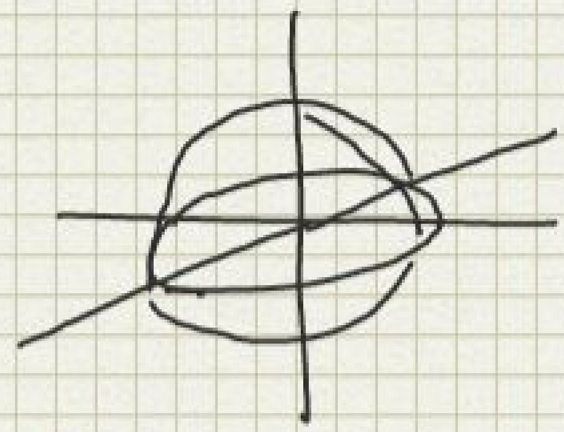
$$\int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=2\sin(\theta)} r\,dr\,d\theta = \int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{2\sin(\theta)} d\theta = \int_0^{\pi/4} \frac{4\sin^2(\theta)}{2}\,d\theta$$

$$= \int_0^{\pi/4} 2\sin^2(\theta)\,d\theta = \int_0^{\pi/4} (1-\cos(2\theta)\,d\theta = \theta - \frac{\sin(2\theta)}{2}\Big|_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{\sin(\pi/2)}{2} = \frac{\pi}{4} - \frac{1}{2}$$

Sphere





$$x^2 + y^2 + z^2 = a^2$$

$$z = \pm\sqrt{a^2 - x^2 - y^2}$$

$$z = \sqrt{a^2 - x^2 - y^2} > 0$$

$$\text{Volume} = 2\iint_D \sqrt{a^2 - x^2 - y^2}\, dA$$

$$= 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta = 2\int_0^{2\pi}\int \sqrt{u}\,\frac{du}{-2}\, dr$$

$$u = a^2 - r^2$$

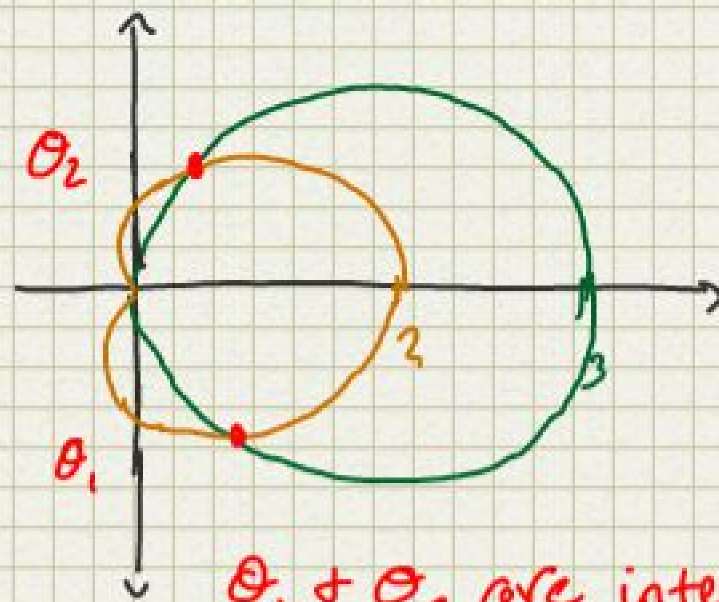
$$\frac{du}{-2} = -2r\,dr$$

$$= 2\int_0^{2\pi} -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\, u^{3/2}\Big|\, d\theta = 2\int_0^{2\pi} -\frac{1}{3}\left(a^2 - r^2\right)^{3/2}\Big|_0^a$$

$$= \frac{-2}{3}\left(0 - a^2\right)^{3/2}\int_0^{2\pi} d\theta = \frac{2}{3}a^3\Big|_0^{2\pi} = \frac{4}{3}\pi a^3$$

$$\iint_D \frac{1}{x}\, dA$$

D: in $R = 3\cos(\theta)$
out $R = 1+\cos(\theta)$



θ_1 + θ_2 are intersection points

$$= \int_{\theta_1}^{\theta_2} \int_{1+\cos\theta}^{3\cos(\theta)} \frac{1}{r\cos(\theta)}\, r\, dr\, d\theta$$

$$\begin{cases} r = 3\cos(\theta) \\ r = 1+\cos(\theta) \end{cases} \Rightarrow \begin{cases} r = 3\cos(\theta) \\ 3\cos(\theta) = 1+\cos(\theta) \end{cases}$$

$$\begin{cases} r = 3\cos(\theta) \\ 2\cos(\theta) = 1 \rightarrow \frac{1}{2} \end{cases} \qquad 2 = 3\cos(\theta)$$

$$\theta = \cos^{-1}\left(\tfrac{1}{2}\right) \begin{cases} \rightarrow 60^\circ \\ \rightarrow -60^\circ \end{cases}$$

$$\int_{-\pi/3}^{\pi/3} \int_{1+\cos(\theta)}^{3\cos(\theta)} \frac{1}{r\cos(\theta)}\, r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{\cos(\theta)}\, r \Big|_{1+\cos(\theta)}^{3\cos(\theta)} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{\cos(\theta)} \left(3\cos(\theta) - 1 - \cos(\theta)\right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (2 - \sec(\theta)\, d\theta = 2\theta - \ln|\sec(\theta) + \tan(\theta)| \Big|_{-\pi/3}^{\pi/3}$$

$$= \ln|\sec(-\pi/3) + \tan(-\pi/3)$$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$

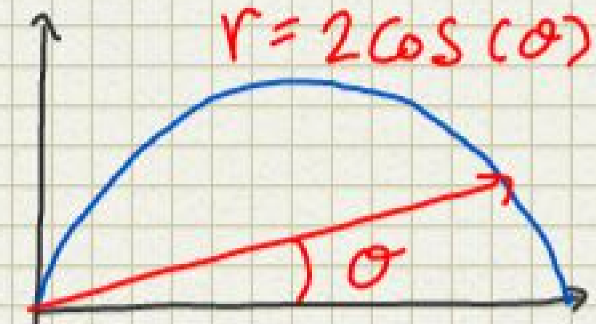
$y=\sqrt{2x-x^2}$

$y^2 = 2x-x^2$

$(x^2-2x+1)+y^2=1$

$(x-1)^2+y^2=1$

$r=2\cos(\theta)$



$$\int_0^{\pi/2} \int_{r=0}^{r=2\cos(\theta)} r\sin(\theta)\sqrt{r^4}\; r\,dr\,d\theta$$

$\sin(\theta)\, r^3$

$u=\cos(\theta)$

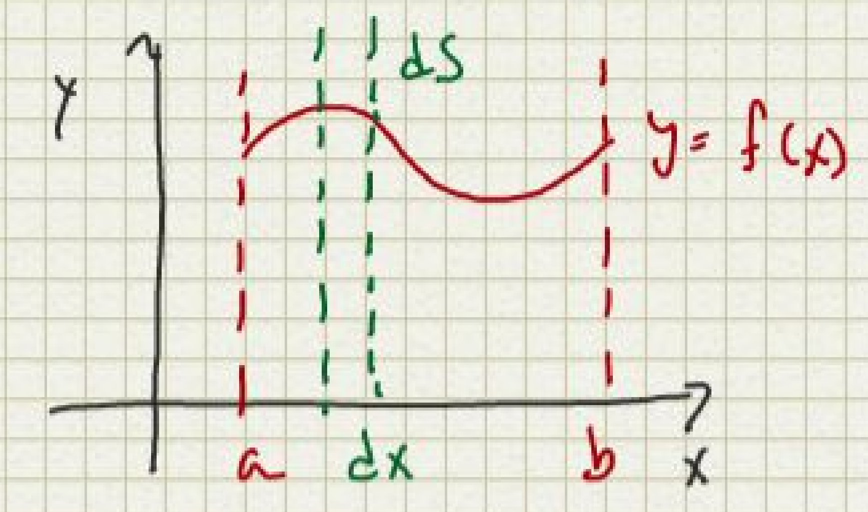
$du=-\sin(\theta)\,d\theta$

$$=\int_0^{\pi/2} \sin(\theta)\,\frac{r^4}{4}\Big|_0^{2\cos(\theta)} d\theta = \int_0^{\pi/2} \sin(\theta)\left(4\cos^4(\theta)-0\right)d\theta$$

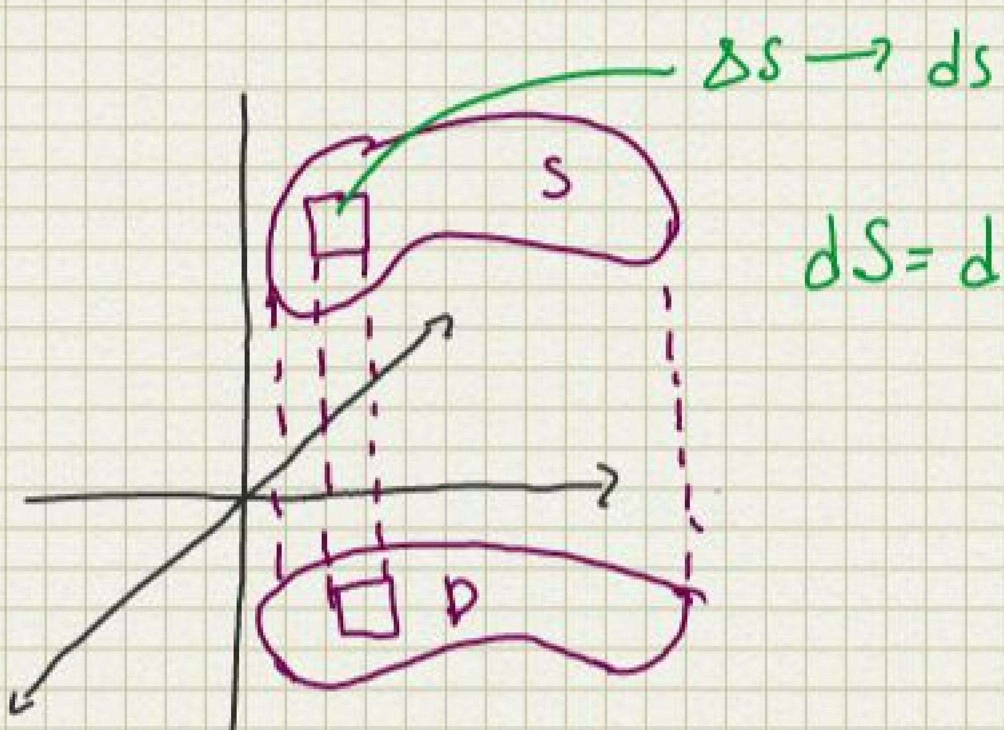
$$=4\int_{---}^{---} -u^4\,du = -4\,\frac{u^5}{5}\Big|_{---}^{---} = \left(-\frac{4}{5}\right)\cos^5(\theta)\Big|_0^{\pi/2}$$

$$=-\frac{4}{5}\left(0-1^5\right)=\frac{4}{5}$$

12.4 surface area



$$S = \int_s ds = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



$$dS = dA \sqrt{1 + f_x^2 + f_y^2}$$

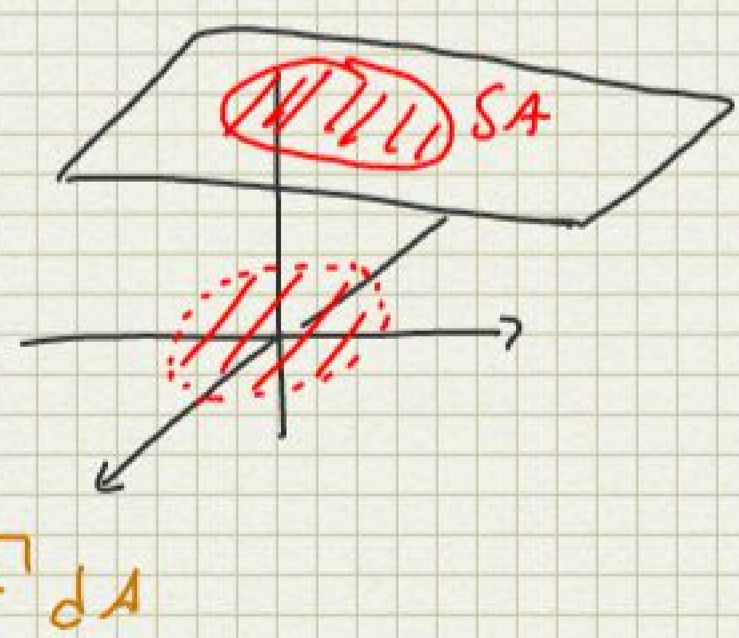
$$SA = \iint_S dS = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Ex

$x + y + z = 4$

Inside

$x^2 + y^2 \leq 4$



$$S = \iint_S ds$$

$$= \iint \sqrt{1 + (-1)^2 + (-1)^2}\, dA$$

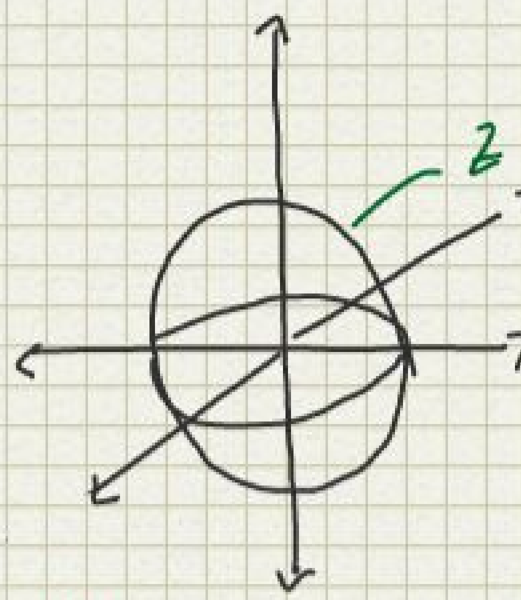
$z = 4 - x - y$ $\qquad \sqrt{3} \iint_D dA$

$z_x = -1$ $\qquad = \sqrt{3}\, \pi (2)^2$

$z_y = -1$ $\qquad = 4\sqrt{3}\pi$

surface area sphere $R = a$

$x^2+y^2+z^2 = a^2$



$z = \sqrt{a^2-y^2-x^2}$

$SA = 2\iint_D \sqrt{1+z_x^2+z_y^2}\, dA$

$\frac{\partial}{\partial x}(\quad) = 2x + 0 + 2z z_x$

$z_x = -\frac{x}{z} \quad z_y = -\frac{y}{z}$

$2\iint_D \sqrt{1+z_x^2+z_y^2}\, dA = 2\iint_D \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\, dA$

$= 2\iint_D \sqrt{\frac{\overbrace{z^2+x^2+y^2}^{\to a^2}}{z^2}}\, dA = 2\iint_D \frac{a}{z}\, dA = 2\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA$

Polar Coordinates

$= 2\int_0^{2\pi}\int_0^a \frac{a^2}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta = 2\int_0^{2\pi}\int_{---}^{---} \frac{a}{\sqrt{u}}\frac{du}{-2}\, d\theta$

$u = 2-r^2$

$-2r\,dr = du$

$= -\int_0^{2\pi} a\, 2u^{1/2}\Big|_{---}^{---} d\theta = -2a\int_0^{2\pi} (a^2-r^2)^{1/2}\Big|_0^a d\theta$

$= 2a\left(0^{1/2}+(a^2)^{1/2}\right)\Big|_0^{2\pi} d\theta = 4\pi a^2$

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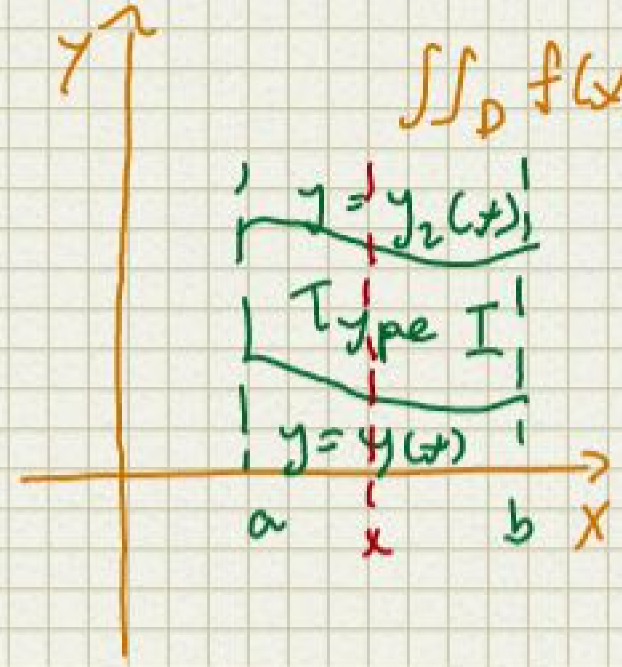
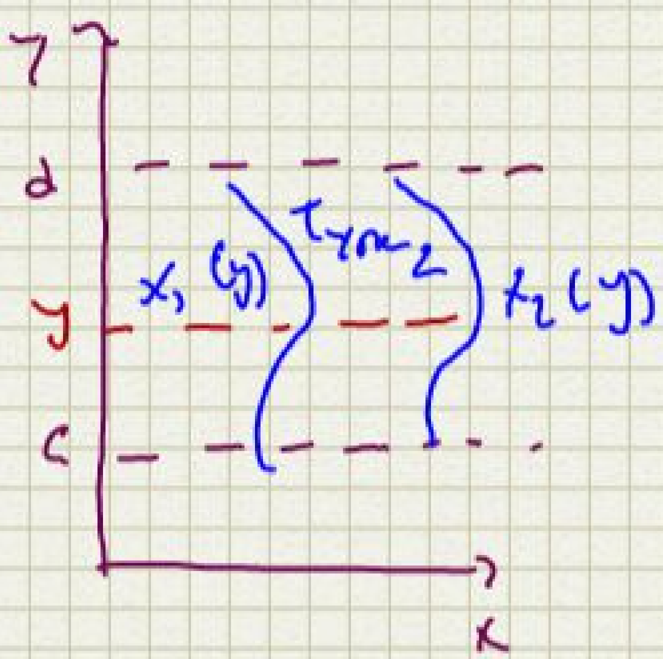
$\iint_D f(x,y)\,dA$

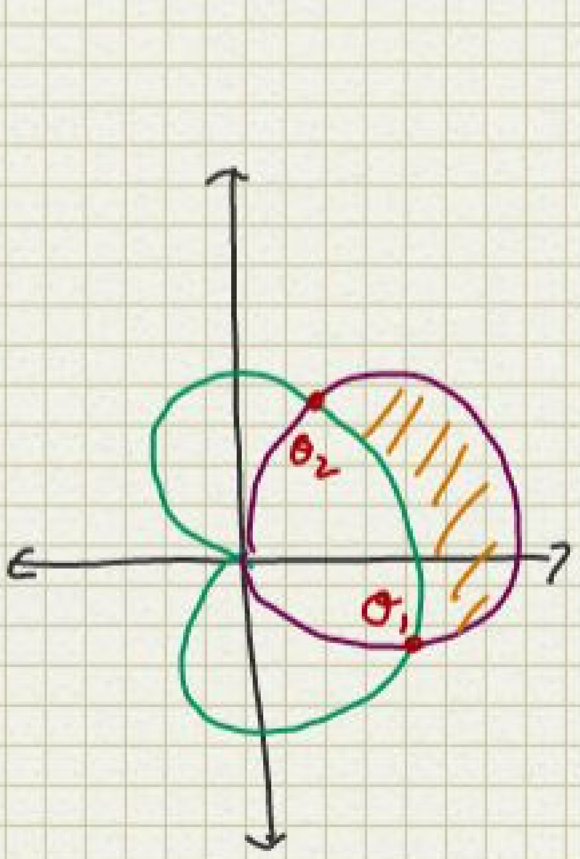


$$\int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\; dy \right) dx$$



$$\int_c^d \int_{x_1,y}^{x_2,y} f(x,y)\,dx \quad dy$$

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \left(\int_{R_1(\theta)}^{R_2(\theta)} f(r,\theta)\, r\, dr \right) d\theta$$



$r = 1 + \cos(\theta)$

$r = 2\cos(\theta)$

$$\int_{\theta_1}^{\theta_2} \int_{r=1+\cos(\theta)}^{2\cos(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

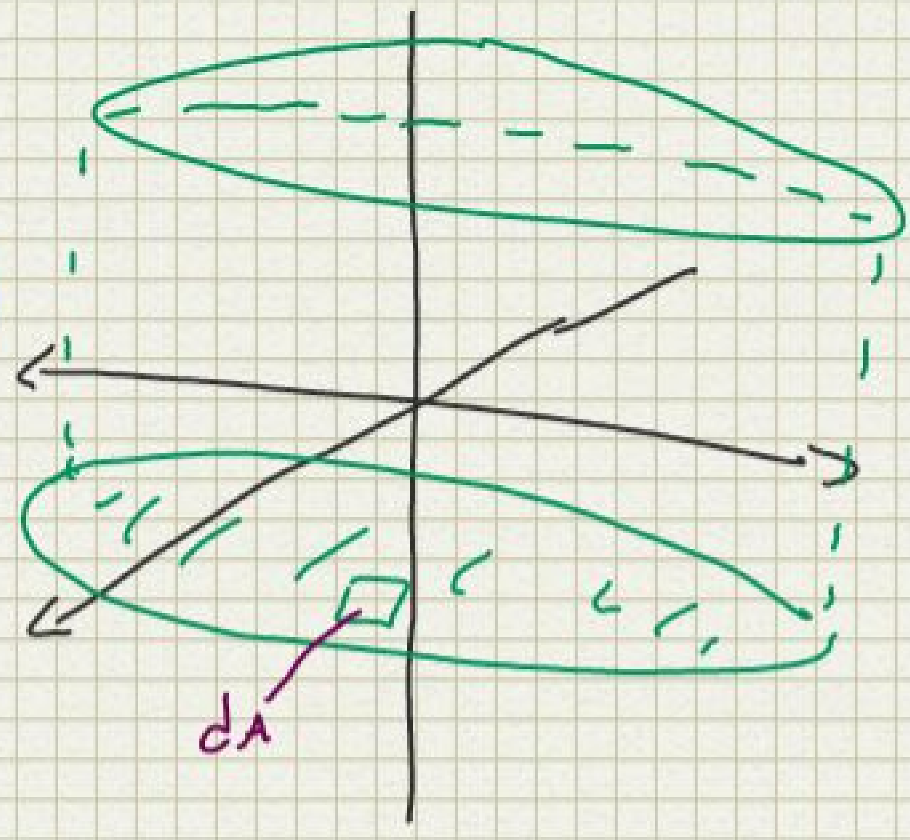
$$\begin{cases} r = 2\cos(\theta) \\ r = 1+\cos(\theta) \end{cases}$$

Surface Area

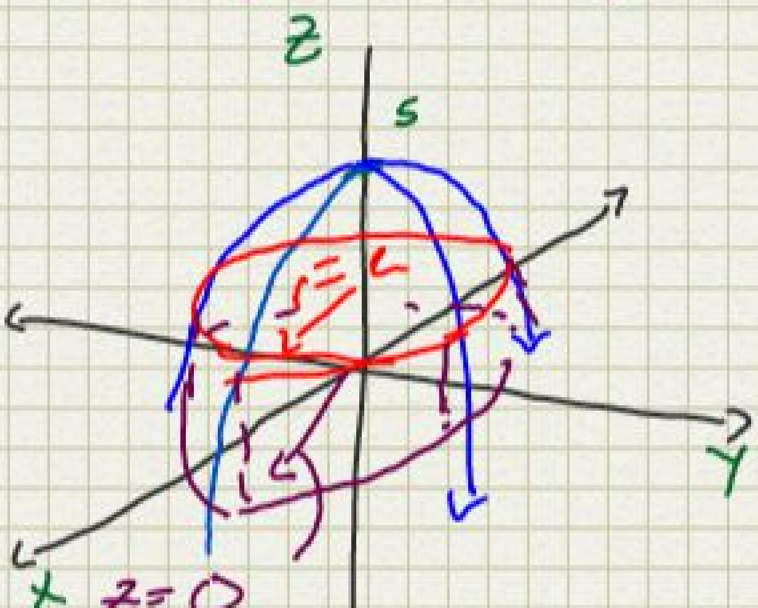
$Z = f(x,y)$ on D

$S = \iint_S dS \quad dS = \sqrt{1 + Z_x^2 + Z_y^2}\, dA$

$= \iint_D \sqrt{1 + Z_x^2 + Z_y^2}\, dA$



$z = f(x,y) = 5 - x^2 - y^2$ (3D)



$z = 0$

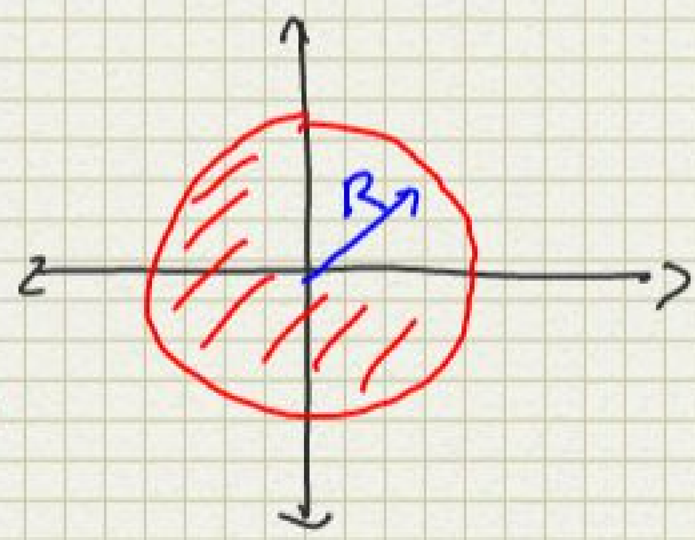
$x^2 + y^2 = 4$

$Z = 5 - x^2 - y^2$

$Z = 1$

$Z = 1$

$x^2 + y^2 = 4$



$S = \iint_S dS = \iint_D \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA$

$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$

$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \int_{---}^{---} \sqrt{u}\, \frac{du}{8}\, d\theta$

$u = 1 + 4r^2$

$du = 8r\, dr$

* $x = r\cos(\theta)$

$y = r\sin(\theta)$

$R_2(\theta) \quad r^2 = x^2 + y$

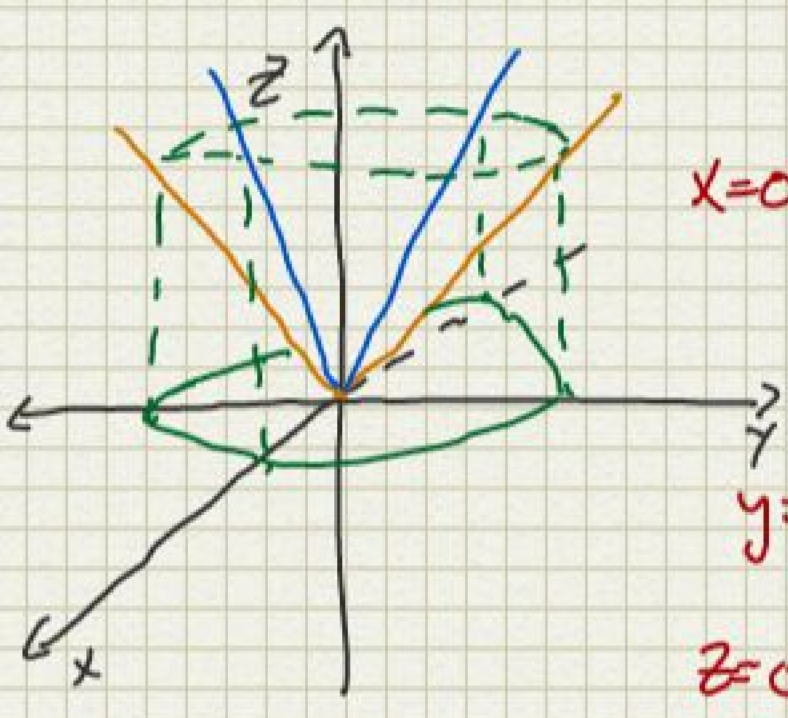
$\tan(\theta) = \frac{y}{x}$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_{---}^{---} d\theta = \frac{1}{12}\int_0^{2\pi} (1+4r^2)^{3/2} \Big|_0^2 d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} (17^{3/2}-1)\, d\theta = \frac{17^{3/2}-1}{12} 2\pi$$

Find the SA of $z = \sqrt{x^2+y^2}$

$0 \le z \le 1$



$z = \sqrt{x^2+y^2}$ cone

$x=0 \begin{cases} z = \sqrt{y^2} \\ z = \pm y \end{cases}$

$y=0 \Rightarrow z = \pm x$

$z=0 \quad \sqrt{x^2+y^2} = 0$

$x=0$

$y=0 \quad z=1$

$x^2+y^2=1 \quad 1 = \sqrt{x^2+y^2}$

$$S = \iint_D \sqrt{1+z_x^2+z_y^2}\, dA$$

2 Methods: Hard & Easy

Hard: $z_x = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \qquad z_x^2 = \frac{x^2}{x^2+y^2}$

$z_y = \ldots$

$z_y^2 = \frac{y^2}{x^2+y^2}$

$$\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{\frac{(x^2+y^2)+x^2+y^2}{x^2+y^2}} = \sqrt{2}$$

$$S = \iint_D \sqrt{2}\, dA = \sqrt{2}\,\pi = \sqrt{2} \iint_D dA = \sqrt{2} \quad \text{Area } r=1$$

$$\frac{\partial}{\partial x}(z^2 = x^2+y^2)$$

$$z_x = -\frac{x}{z} \quad z_y = -\frac{y}{z} \quad = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{z^2+(x^2+y^2)}{z^2}} = \sqrt{\frac{2\cancel{z^2}}{\cancel{z^2}}}$$

$$= \sqrt{2}$$

$f(x,y,z)$

$$\iiint_D f(x,y,z)\,dV$$

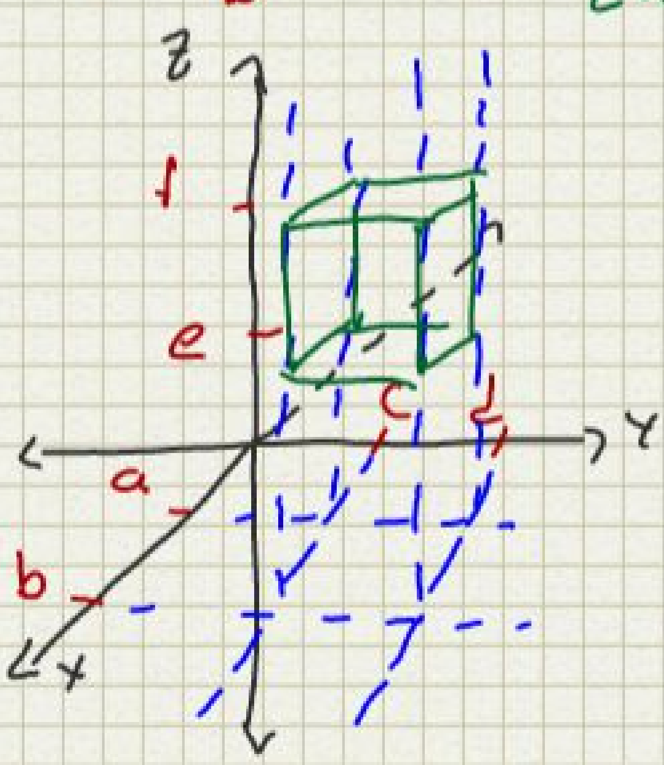
6 possible iterated integrals

2 ways

$$\int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

$$\int_a^b \int_e^f \int_c^d f(x,y,z)\,dy\,dz\,dx$$

$a \le x \le b$
$c \le y \le d$
$e \le z \le f$



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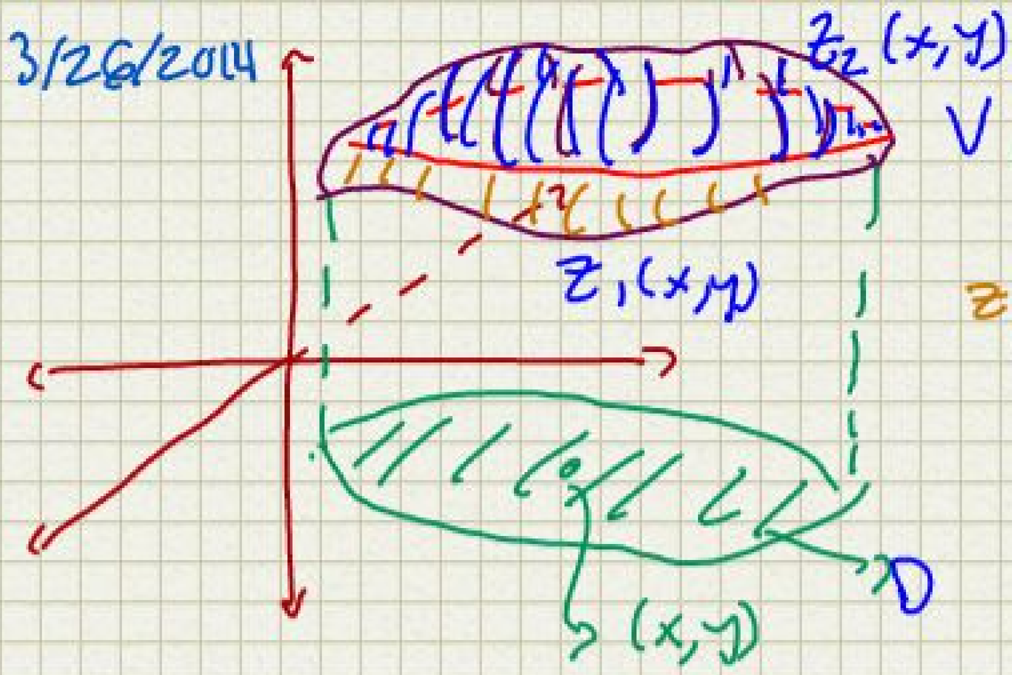


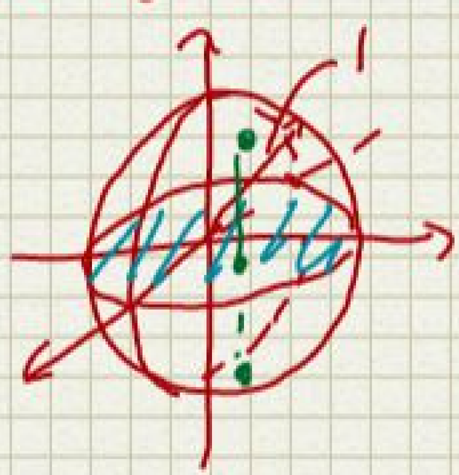
V is z-simple integraly if

$z_1(x,y) \le z \le z_2(x,y)$

$$\iiint_V f(x,y,z) = \iint_D \underbrace{\int_{z_1}^{z_2} f(x,y,z)\,dz}_{h(x,y)}\,dA$$

$$= \iint_D h(x,y)\,dA$$

$V: x^2+y^2+z^2=1$

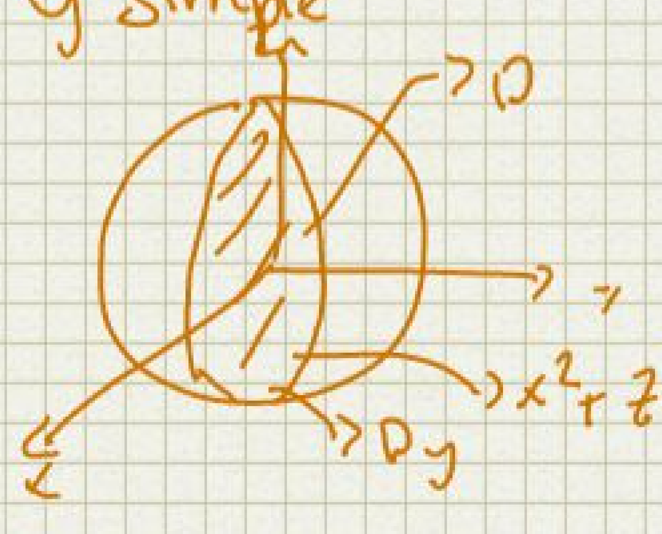


z simple:

$$\iiint_V f(x,y,z)\,dv$$

$$= \iint_D \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dA$$

y simple



$$\iint_{D_y} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dA$$

$y = \pm\sqrt{1+x^2-y^2}$

x-simple region

$$= \iint_{D_1} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dA \qquad dA = dy \cdot dz$$

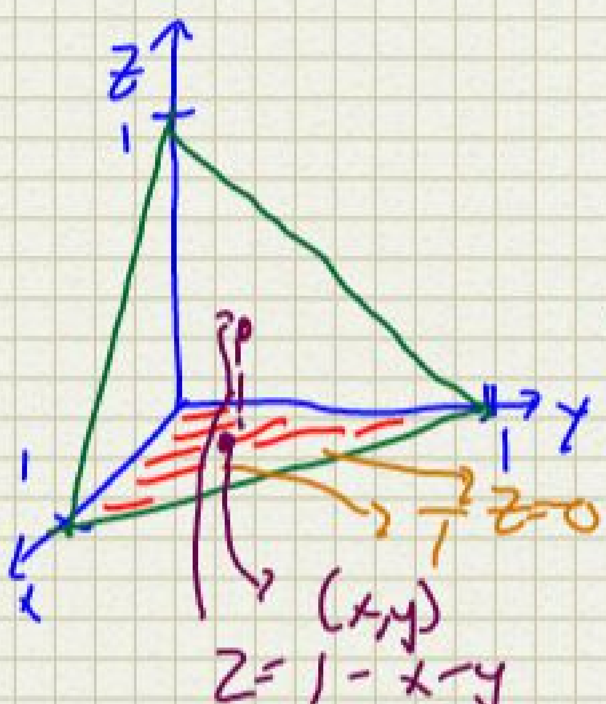
Ex

$V = \iiint_V 1\,dV$ V is region ~~bounded~~ enclosed by

$x=0 \quad x+y+z=1$

$y=0$

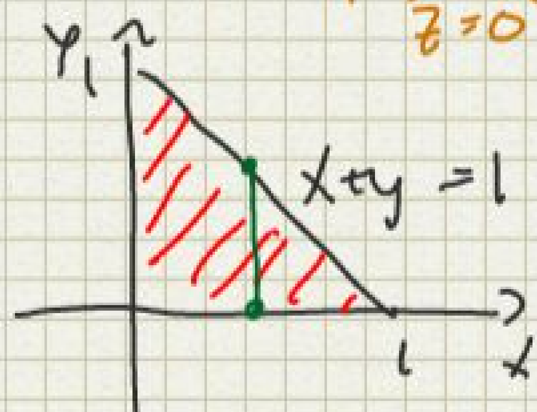
$z=0$



$z=y=0$, $x=1$ $\qquad$ $y=x=0$, $z=1$ $\qquad$ $z=0=x$, $y=1$

$$\frac{A_T \cdot 1}{3}$$

$$\text{Volume} = \iint_T \int_{z=0}^{z=1-x-y} 1\,dz\,dA = \iint_T z\Big|_0^{1-x-y} dA = \iint_T (1-x-y)\,dA$$



$$= \int_0^1 \int_{y=0}^{y=1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 (1-x)y - \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x)(1-x) - \frac{(1-x)^2}{2}\,dx = \frac{1}{2}\int (1-x)^2\,dx$$

$$= -\frac{1}{2}\left(\frac{(1-x)^3}{3}\right)\Big|_0^1$$

$$= -\frac{1}{6}(0-1) = \frac{1}{6}$$

3/27/14

Integration in 3D

$$\iiint_V f(x,y,z)\,dV = \iint_D \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) = \iint_D g(x,y)\,dA$$

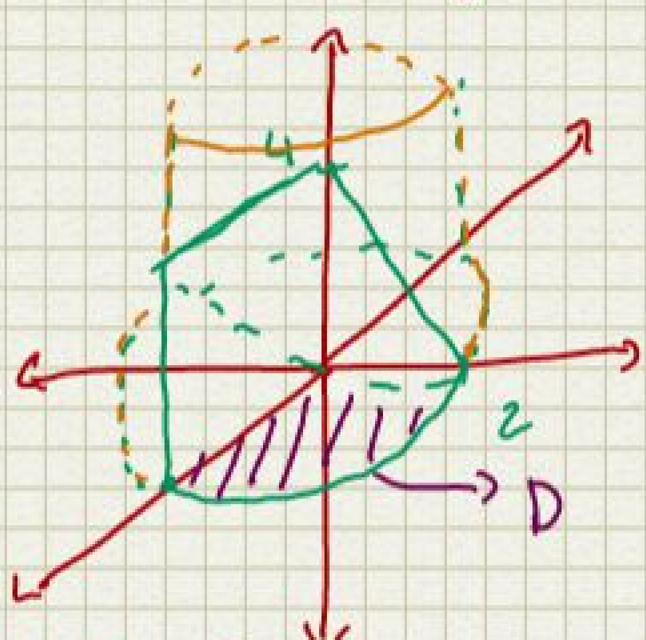
V is simple for every point w/ x,y coordinate

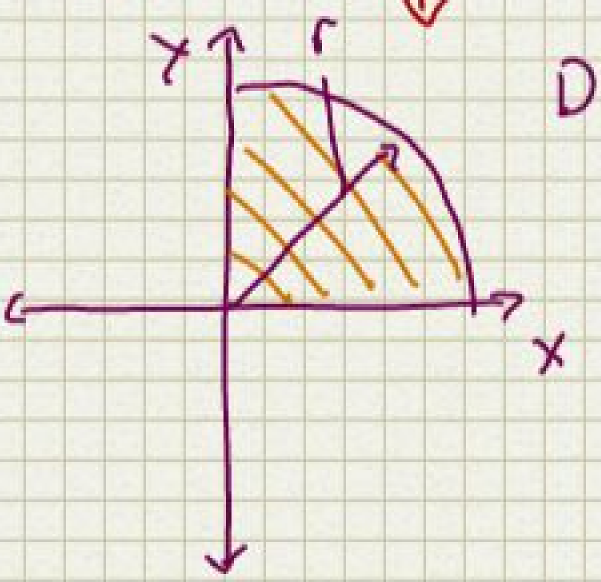
$\iiint_V dV = V$ has upper & lower limits at $z=0$, $z=f(x,y)$

$$\iiint_V x\,dV$$

inside cylinder $x^2+y^2=4$
below cylinder $2y+z=4$

$V:\ \left.\begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix}\right\}$ first octant





$$\iiint_V = \iint_D \int_{z=0}^{z=4-2y} x\,dz\,dA$$

(Convert to Polar)

$$\int_0^{\pi/2}\int_0^2 r\cos(\theta)(4-2r\sin(\theta))r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^2 4r^2\cos(\theta)-2r^3\sin(\theta)\cos(\theta))\,dr\,d\theta$$

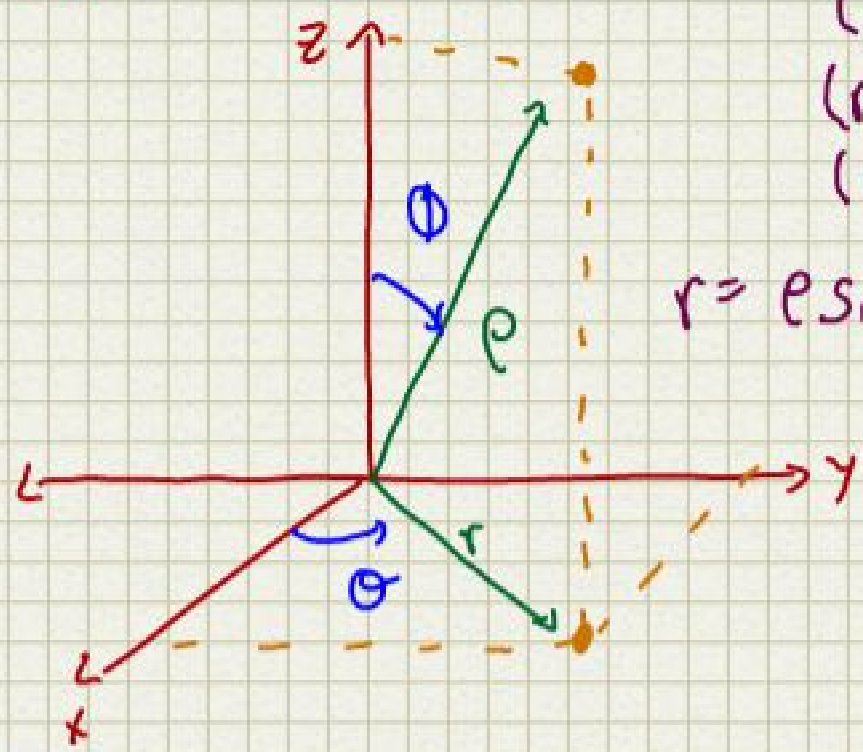
$$=\int_0^{\pi/2} \frac{4}{3}r^3\cos(\theta)\Big|_0^2 - \frac{r^4}{4}\sin(2\theta)\Big|_0^2\,d\theta$$

$$=\int_0^{\pi/2} \frac{32}{3}\cos(\theta)-4\sin(2\theta)\,d\theta$$

$$=\frac{32}{3}\sin(\theta)\Big|_0^{\pi/2}+\frac{4}{2}\cos(2\theta)\Big|_0^{\pi/2} = \frac{32}{3}-4$$

$$=20$$

Spherical Coordinates



(x, y, z) - rectangular
$(r, \theta\ z)$ - cylindrical
(ρ, θ, ϕ) - Spherical

$r = \rho \sin(\theta)$

Conversions

Cylin $\rightarrow$ spherical

$$\begin{cases} r = \rho \sin(\phi) \\ \theta = \theta \\ z = \rho \cos(\phi) \end{cases}$$

Rec $\rightarrow$ Spherical

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases}$$

Spherical $\rightarrow$ Cylindrical

$$\begin{cases} \rho^2 = r^2 + z^2 \\ \theta = \theta \\ \tan(\phi) = \frac{r}{z} \end{cases}$$

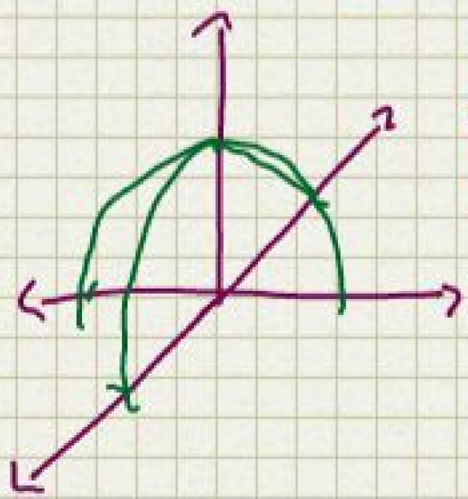
Spherical $\rightarrow$ Rec

$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\phi) = \frac{\sqrt{x^2+y^2}}{z} = \frac{r}{z}$$

$$\cos(\phi) = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



$$\begin{cases} z = 4 - 4x^2 - 4y^2 \\ z = 0 \end{cases}$$ Volume: $\iiint_V dV$

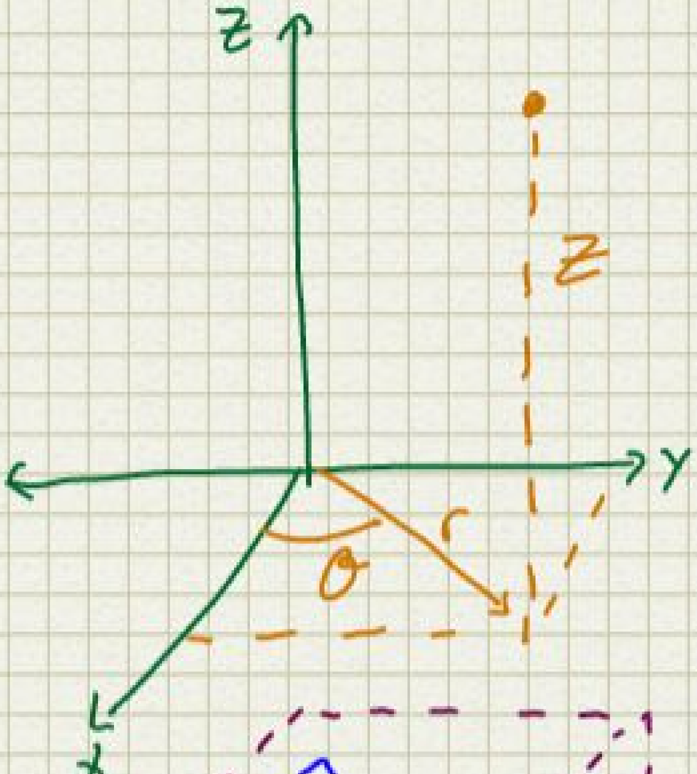
$$V = \iint_D \int_0^{4-4x^2-4y^2} dz\, dA = \iint_D (4 - 4x^2 - 4y^2)\, dA$$

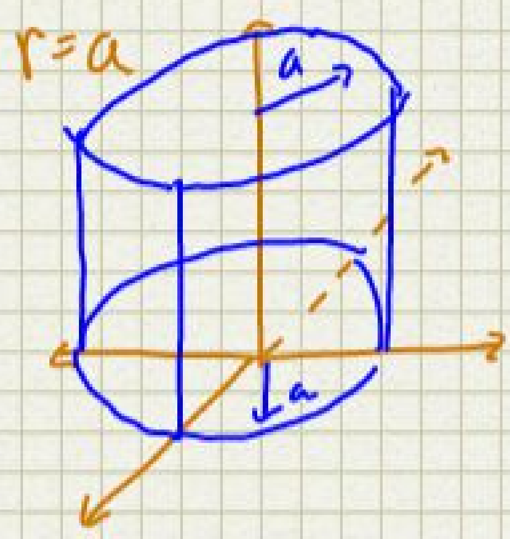
$$= \int_0^{2\pi} \int_0^1 (4 - 4r^2)\, r\, dr\, d\theta = \int_0^{2\pi} 2r^2 - r^4 \Big|_0^1 d\theta$$

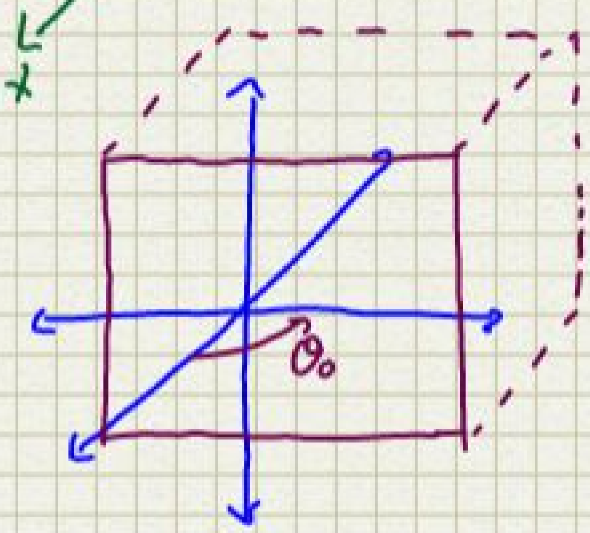
$$= \theta \Big|_0^{2\pi} = 2\pi$$

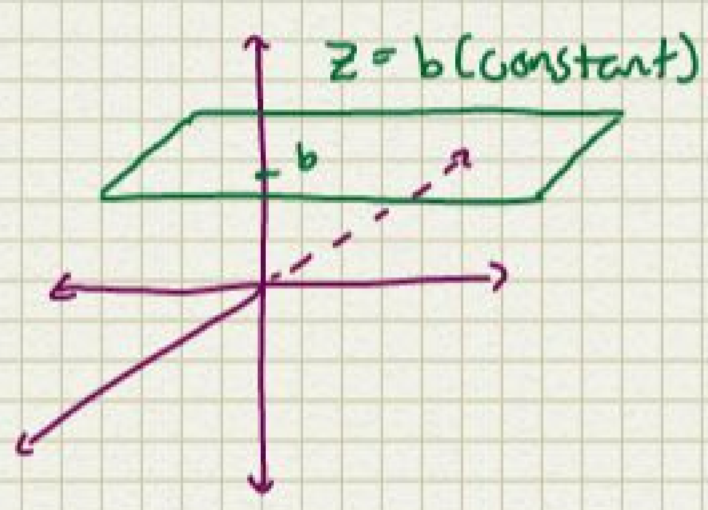
3D Cylindrical Coordinates

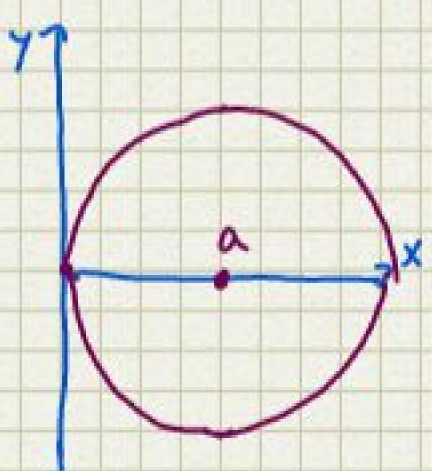
$$\text{rec}\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} \quad \text{Cylin}\begin{cases} r \\ \theta \\ z \end{cases} \Rightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan(\theta) = \frac{y}{x} \\ z = z \end{cases}$$

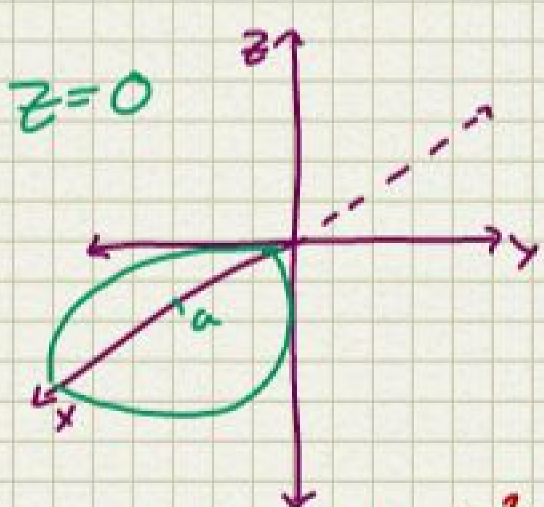












$$x^2 + y^2 + z^2 = a^2 \Longleftrightarrow r^2 + z^2 = a^2 \qquad z^2 = x^2 + y^2$$

$$z = ax^2 + ay^2 + c \Longleftrightarrow z = ar^2 + c \qquad z = \pm\sqrt{x^2+y^2}$$

$$dV = dz\,dA = dz\,r\,dr\,d\theta = r\,dz\,dr\,d\theta$$

Always integrate like:

$$\iiint_V f(r,\theta,z)\,dV = \underbrace{\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}}_{D}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$

Find Volume:

$$V:\begin{cases}\text{above } z = 4-4x^2-4y^2 \\ \text{below } z=0\end{cases} = \begin{cases} z = 4-4r^2 \\ z = 0\end{cases}$$

$$V = \int_0^{2\pi}\int_0^1\int_{z=0}^{z=4-4r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,z\Big|_0^{4-4r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4-4r^3)\,dr\,d\theta = \ldots$$

Volume?

$x \geq 0$ $\quad x^2+y^2=2y \quad\quad x^2+(y^2-2y+1)=1$

$y \geq 0 \quad\quad\quad\quad\quad\quad\quad\quad x^2+(y-1)^2=1$

$z \geq 0$ $\quad z=\sqrt{x^2+y^2} \quad\quad r = 2\sin(\theta) \quad\quad z=r, \; z=0$

$$V = \int_0^{\pi/2}\int_{r=0}^{r=2\sin(\theta)}\int_{z=0}^{z=r} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\sin(\theta)} r\,z\Big|_0^{r} dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{2\sin(\theta)} d\theta = \frac{8}{3}\int_0^{\pi/2}\sin^3(\theta)\,d\theta = \frac{8}{3}\int_0^{\pi/2}\left(\sin(\theta)-\sin(\theta)\cos^2(\theta)\right)d\theta$$

$$= \frac{8}{3}\left(-\cos(\theta)\Big|_0^{\pi/2} - \int_{---}^{---} u^2\,du\right) = \frac{16}{9}$$

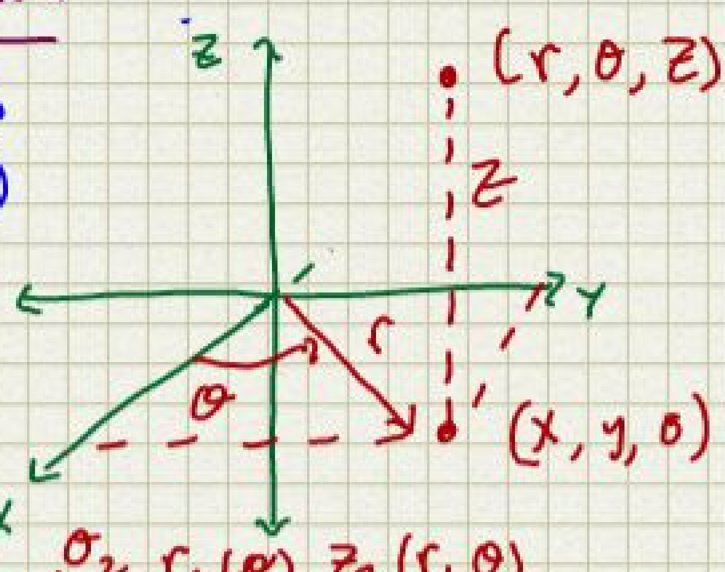
$u = \cos(\theta)$

$du = -\sin(\theta)\,d\theta$

4/1/14

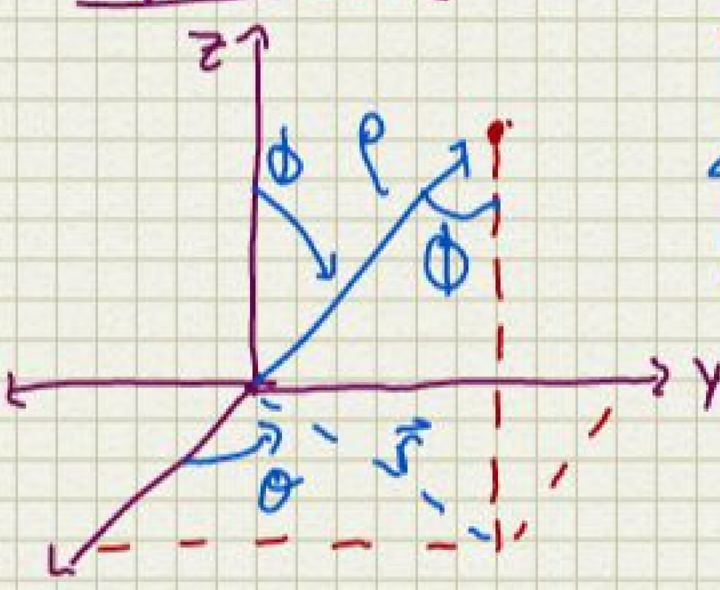
Cylindrical

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$



$$\iiint_V f\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

Spherical

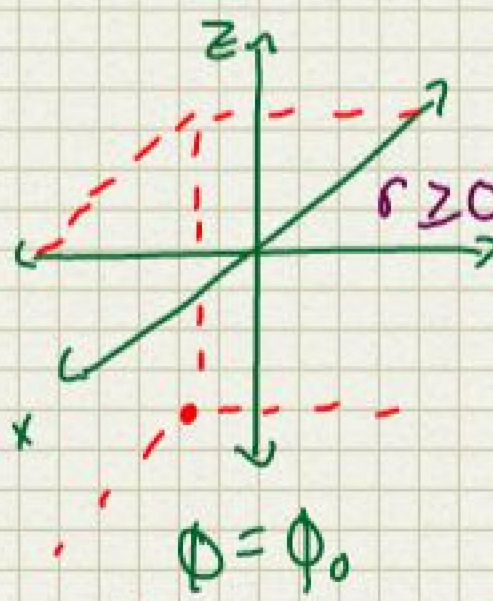


$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$

$\rho^2 = r^2 + z^2 = \underline{x^2 + y^2 + z^2}$

$\tan(\theta) = \frac{y}{x}$

$\cos(\theta) = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$



$\pi \le \theta \le \frac{3}{2}\pi$

$r \ge 0$ $\rho \ge 0$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$x, y \,\&\, z \le 0$

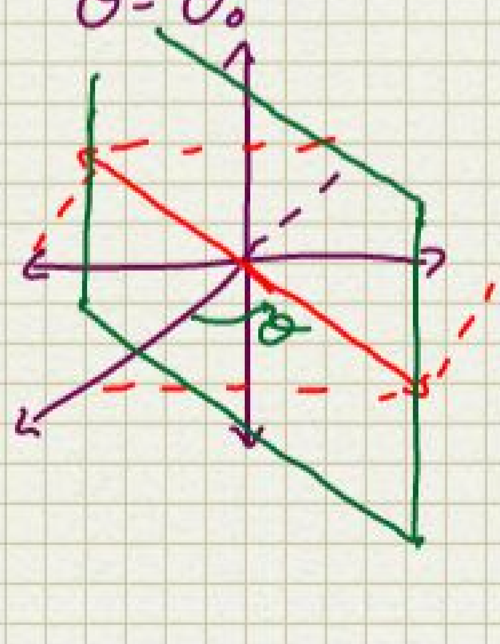
$\phi = \phi_0$

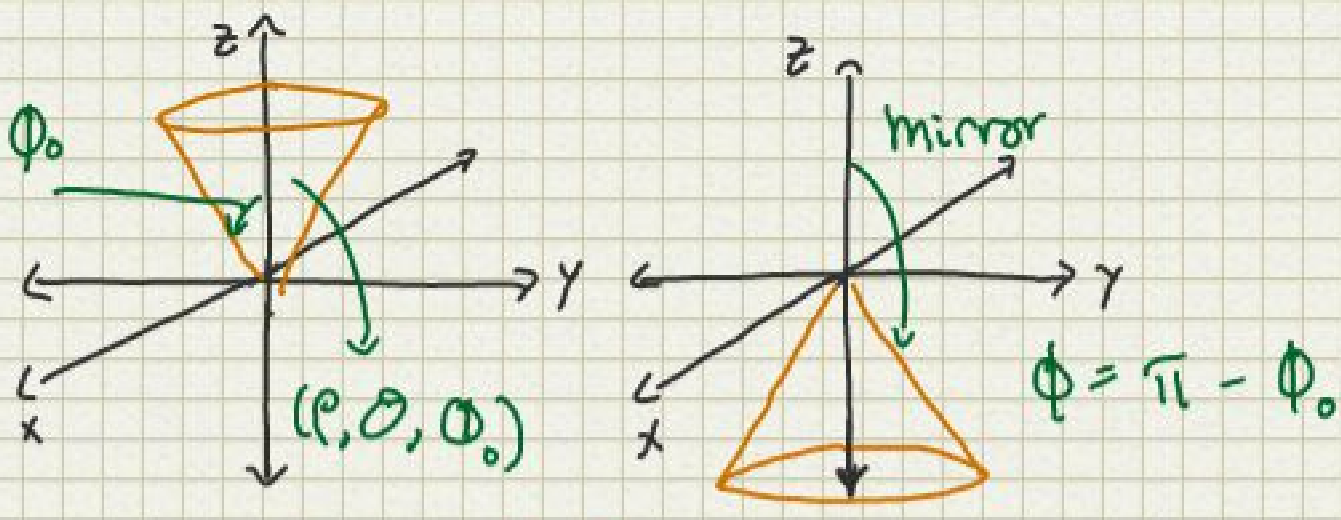
for all θ & ρ

$\rho = a$ $\rho^2 = a^2 = \underline{x^2 + y^2 + z^2}$

sphere

$\theta = \theta_0$





$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{4}} \int_{\rho=1}^{\rho=2} \frac{1}{\rho}\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin(\phi)\, d\phi \int_1^2 \rho\, d\rho = 2\pi(2)\left(\frac{\rho^2}{2}\Big|_1^2\right)$$

$$= 2\pi\left(2^2 - 1^2\right) = 6\pi \text{ units}^3$$

4/2/2014

12.8 Jacobian

Polar: $dx\,dy\ r\,dr\,d\theta$

Cylin: $dx\,dy\,dz = r\,dz\,dr\,d\theta$

sphere: $dx\,dy\,dz = \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta$

2D

if $x(u,v)$ & $y(u,v)$ then Jacobian is equal to

$$\text{abs}\left(\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right)$$

Polar Coordinates

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$$\text{Jac.} = \text{abs}\left(\begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}\right)$$

$$= \text{abs}\left| r\cos^2(\theta) + r\sin^2(\theta) \right|$$

$$= \text{abs}(r) = r$$

$z = z$

Jacobian of transformation

$$J = \text{abs}\left(\begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}\right) = \text{abs}\left(\begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 0 \end{vmatrix}\right)$$

$$= \text{abs}(\cos^2(\theta) r + \sin^2(\theta) r) = \text{abs}(r) = r$$

$x = \rho\sin(\phi)\cos(\theta)$

$y = \rho\sin(\phi)\sin(\theta)$

$z = \rho\cos(\theta)$

$$J = \text{abs}\left(\begin{vmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\theta) & 0 & -\rho\sin(\phi) \end{vmatrix}\right)$$

$= \text{abs}(\sin(\phi)\cos(\theta)(-\rho^2\sin^2(\phi)\cos(\theta)) - \rho\sin(\phi)\sin(\theta)(-\rho\sin^2(\phi)$
$\sin(\theta - \rho\cos^2(\phi)\sin(\theta))$
$+ \rho\cos(\phi)\cos(\theta)(-\rho\sin(\theta)\cos(\phi)\cos(\theta)$

$= \rho^2\sin(\phi)(\cos^2(\theta)\sin^2(\phi) + \sin^2(\theta) + \cos^2(\phi)\cos^2(\theta)$

$\rho^2\sin(\phi) = (\cos^2(\theta)(\sin^2(\phi) + \cos^2(\phi) + \sin^2(\theta)) = 1$

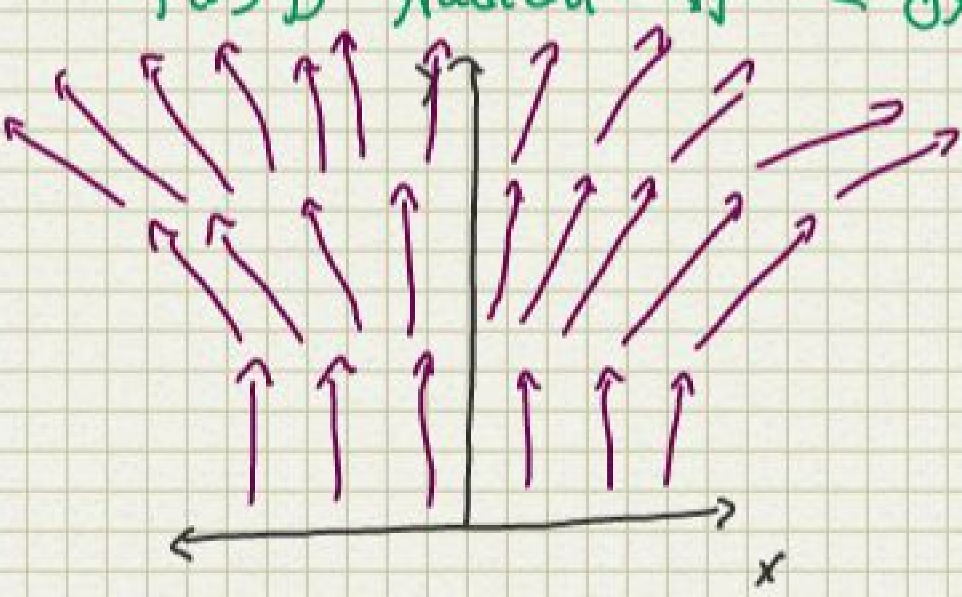
Spherical $dx\,dy\,dz \to \rho^2\sin(\phi)$

Test Ch12.1 → 12.7

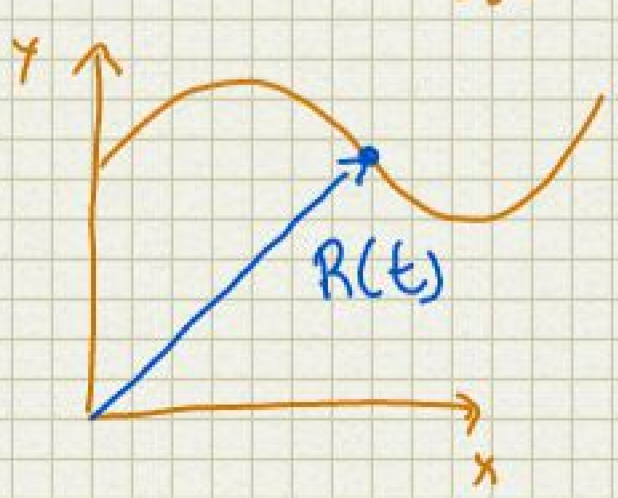
Ch13

Vector Fields 13.1

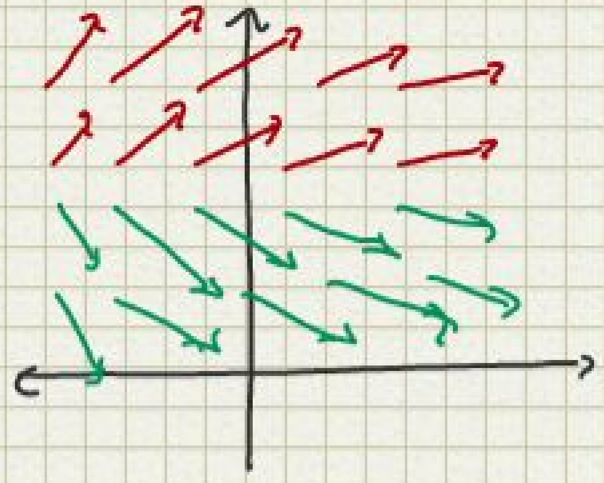
$f(x,y)$ gradient $= \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle g_1(x,y), g_2(x,y) \rangle$



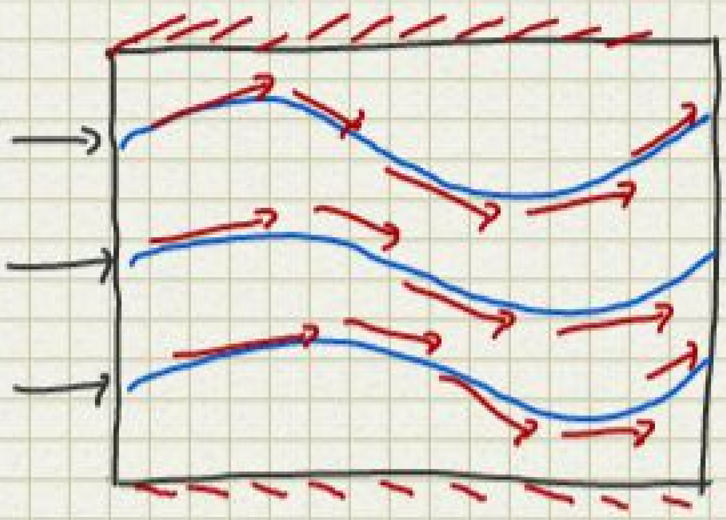
$\vec{R}(t) = \langle g_1(t), g_2(t) \rangle$ $\vec{F}(x,y) = \langle \sin(x+y), e^4 \rangle$ Vector field



$g_1(t) = x(t)$
$g_2(t) = y(t)$



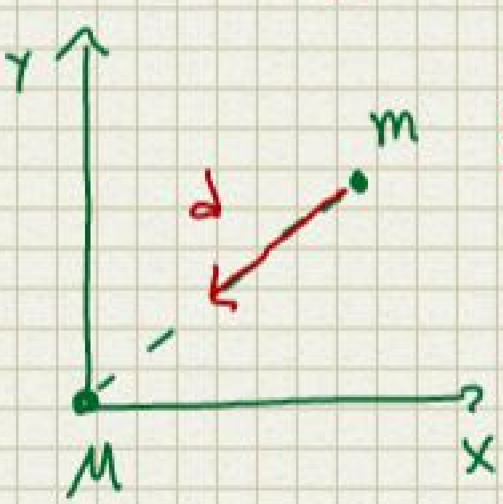
vector field is used in fluid mechanics
+ electric vetor field



$\vec{V} = \langle V_1(x,y), V_2(x,y) \rangle$

Gravitational Vector Field

$\vec{F} = \frac{GMm}{d}$



$\vec{F} = \frac{GMm}{d} \frac{\langle -x,-y \rangle}{\sqrt{x^2+y^2}} = GMm \frac{\langle -x,-y \rangle}{(y^2+x^2)^{3/2}}$

Vector Field F is a quantity define as vector

$\vec{F} = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$\vec{F}(x,y,z) = \langle x^2+z, y+x, z^2 \rangle$

$\vec{F}(0,1,0) = \langle 0,1,0 \rangle$

Domain of $\vec{F}$ is the largest set for f_1, f_2 + f_3 make sense/are defined

Divergent & Curl

-derivatives of vector field

Div: $\text{div}\,\vec{F} = \nabla \cdot \vec{F}$ $\qquad \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle f_1, f_2, f_3 \rangle$

$= \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3 = g(x, y, z)$

Curl: $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(f_3\left(\frac{\partial}{\partial y}\right) - f_2\left(\frac{\partial}{\partial z}\right)\right)\hat{i} - \left(f_3\left(\frac{\partial}{\partial x}\right) - f_1\left(\frac{\partial}{\partial z}\right)\right)\hat{j}$$
$$+ \left(f_2\left(\frac{\partial}{\partial x}\right) - f_1\left(\frac{\partial}{\partial y}\right)\right)\hat{k}$$

13.1 Vector Field 4/3/2014

$\vec{F}(x,y,z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$\text{div}\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \vec{F}$

$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ $(mF + mG)$
$= m\langle f_1, f_2, f_3 \rangle + m\langle g_1, g_2, g_3 \rangle$

$\vec{F} \cdot \vec{G} = f_1 g_1 + f_2 g_2 + f_3 g_3$

$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$

$\nabla \times (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \times \vec{G} + \vec{F} \times (\nabla \times \vec{G})$

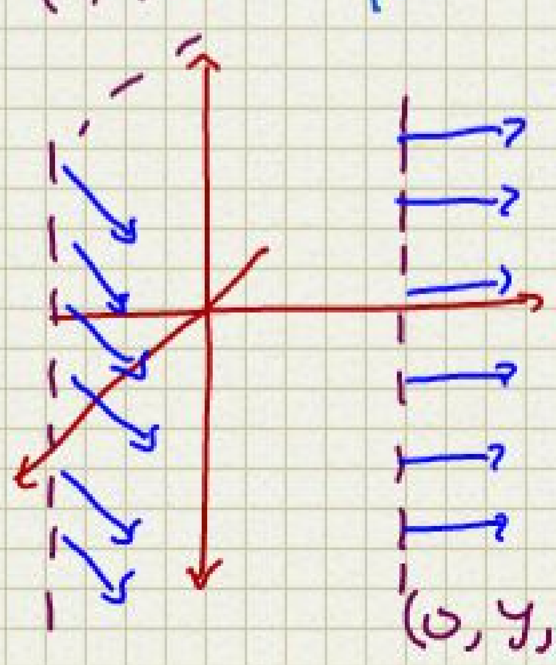
Ex $\vec{F} = \langle \sin(x), e^y, \sin(z) \rangle$

$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\right)\sin(x) + \left(\frac{\partial}{\partial y}\right) e^y + \left(\frac{\partial}{\partial z}\right)\sin(z)$ $\nabla \times \vec{F} = \vec{0}$
$= \cos(x) + e^y + \cos(z)$

Ex $\vec{F} = \langle y^2 z, e^{xz}, \sin(xy) \rangle$
$f(y,z)$ $f(x,z)$ $f(x,y)$

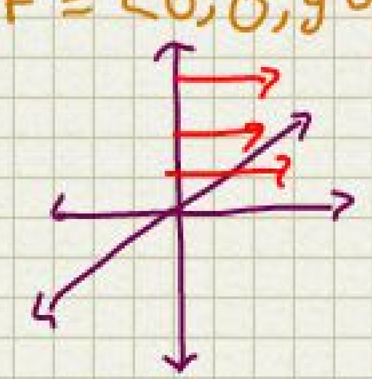
$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & e^{xz} & \sin(xy) \end{vmatrix} = \hat{i}(\cos(xy)(x) - xe^{xz}) - \hat{j}(y\cos(xy) - y^2)$
$+ \hat{k}(ze^{xz} - 2yz)$

$(x,0,z)$

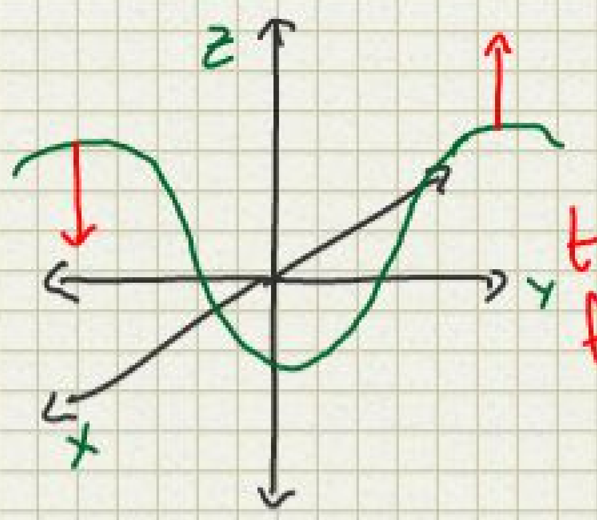


$(0,y,z)$

2D vector in space
$\vec{F} = \langle f_1(x,y), f_2(x,y) \rangle = \langle xy, y^2 + x^2, 0 \rangle$
$\nabla \times \vec{F} = \langle 0, 0, g(x,y) \rangle$ $\nabla \times \vec{F} = \langle 0, 0, x \rangle$



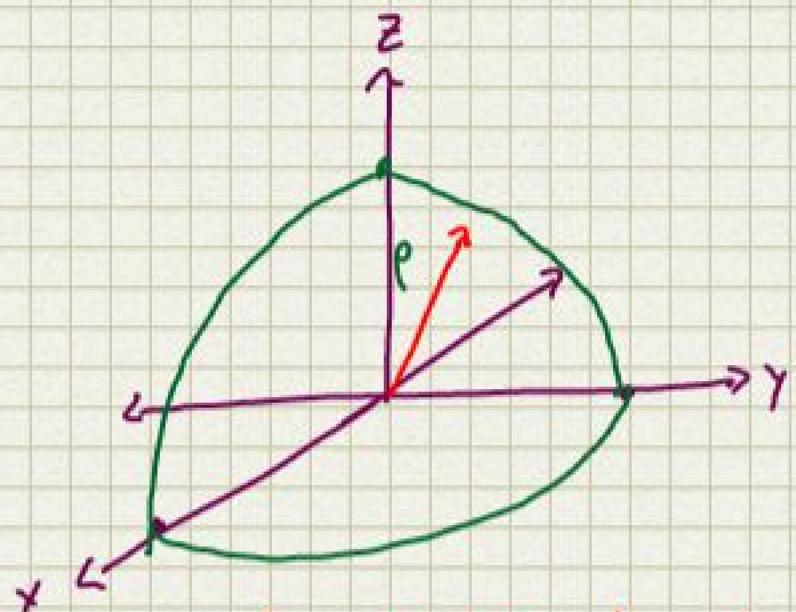
$\nabla \times \vec{F} = 0$



tangent to function

Volume;

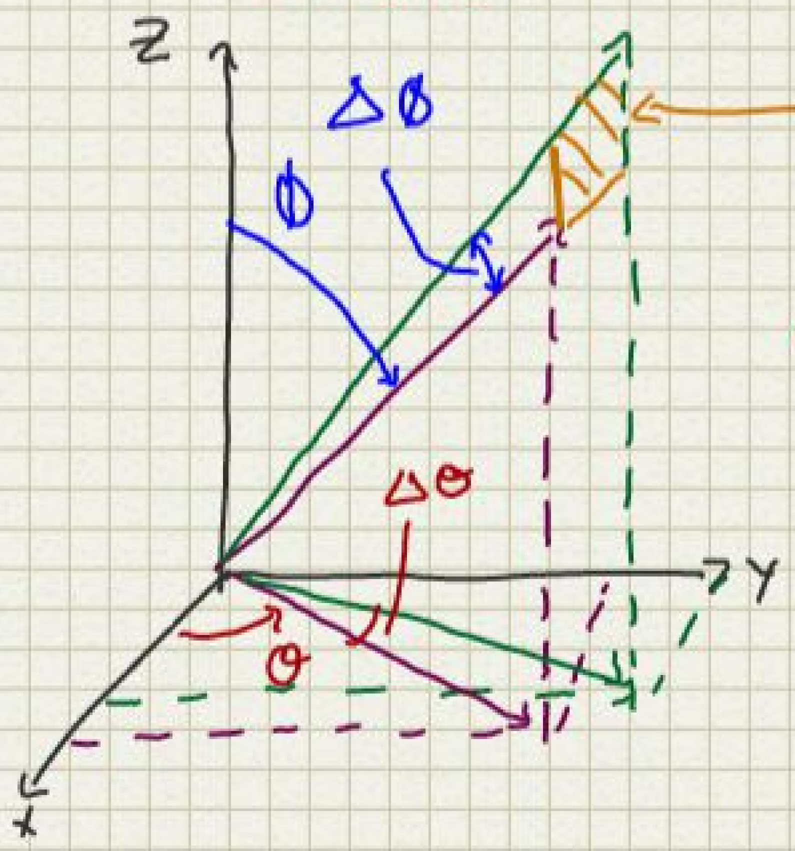
$$\begin{cases} x^2+y^2+z^2 \le a^2 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases}$$



$$\text{volume} = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} dV$$

$L \cdot \phi \cdot \theta \neq$ volume

$$\text{volume} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} dV \qquad dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$



$$\Delta V = \left[(\Delta\rho)(\rho \sin(\phi))\right]\left[\Delta\theta\rho\right]\left[\Delta\phi\right]$$

$$\text{Volume} = \iiint f(\rho, \theta, \phi)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Ex: Sphere

radius = 8

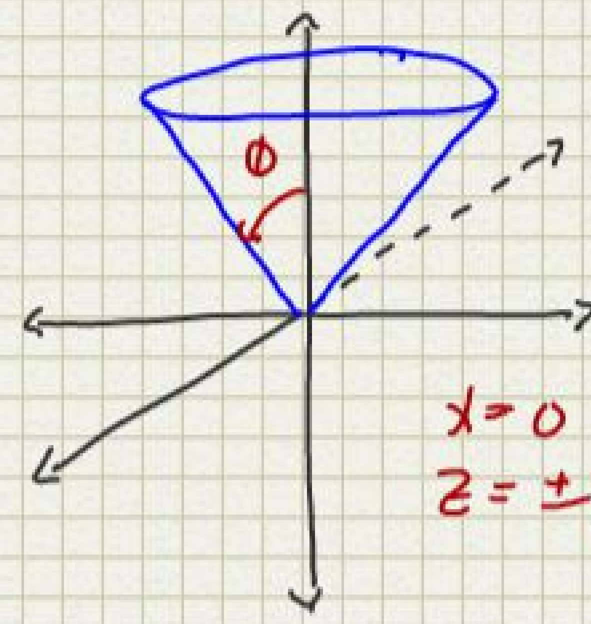
$$V=\iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{a} \rho^2 d\rho = (2\pi)\left(-\cos(\phi)\Big|_0^{\pi}\right)\left(\frac{1}{3}\rho^3\Big|_0^{a}\right)$$

$$= \frac{4}{3}\pi a^3$$

HW: Cone

$$V=\iiint_V \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV$$



$\phi = 45°$

Inside the Cone: $z=\sqrt{x^2+y^2}$

Inside: $x^2+y^2+z^2=4$

Outside: $x^2+y^2+z^2=1$

$x=0$

$z=\pm y$

$$\text{Volume} = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$\int_0^{\pi/2}\int_0^{\pi/4} \sin(\phi)\, \frac{\rho^3}{3}\Big|_0^{\cos(\phi)} d\theta\, d\phi = \frac{1}{3}\int_0^{\pi/2}\int_0^{\pi/4} \sin(\phi)\cos^3(\phi)\, d\theta\, d\phi$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin(\phi)\cos^3(\phi)\, d\phi \int_0^{\pi/4} d\theta = \frac{1}{3}\left(\frac{\pi}{4}\right)\int u^3$$

$u = \cos(\phi)$

$du = -\sin(\phi)\, d\phi$

$$= -\frac{\pi}{12}\left(\frac{\cos(\phi)^4}{4}\right)\Big|_0^{\pi/2} = \frac{\pi}{48}$$

4/9/14

$\vec{F} = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{F} \cdot \vec{T}$

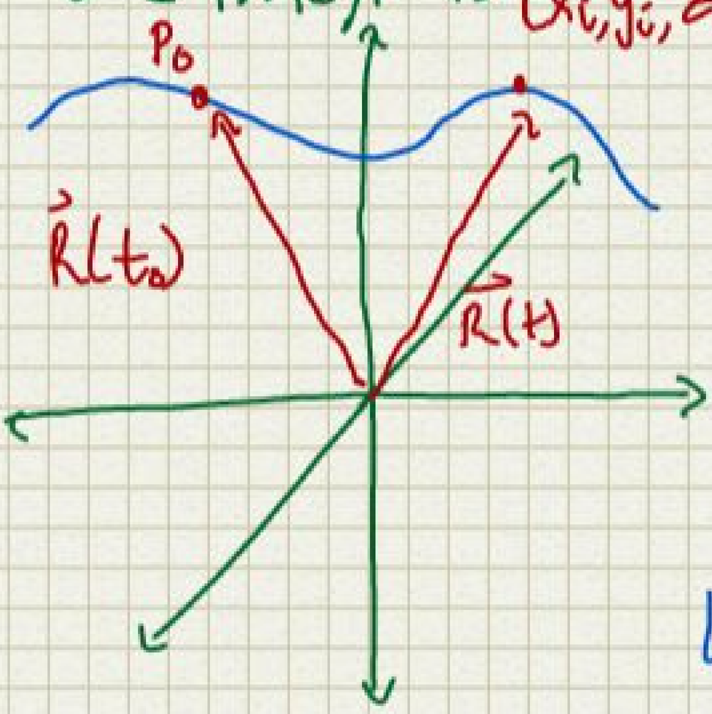
$\vec{F} \cdot \vec{G}$

$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \vec{F}$

$\nabla \times \vec{F} = \langle$

13.2

Line integrals (x_i^*, y_i^*, z_i^*)



$R(t) = \langle x(t), y(t), z(t) \rangle$

$$\lim_{m \to \infty} \sum_{i=1}^{m} \Delta s$$

$$S = \int_{s_0}^{s_1} ds = \int_{t_0}^{t_1} \|\vec{R}'(t)\| dt$$

$$= \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Line I of $f(x,y,z)$ on $\vec{R}$

$$L.I. = \int_{s_0}^{s_1} f(x,y,z)\, ds = \lim_{m \to \infty} \sum_{i=1}^{m} f(x_i^*, y_i^*, z_i^*) \Delta s$$

$$= \int_{t_0}^{t_1} f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_{t_0}^{t_1} f(\vec{R}(t)) \|\vec{R}'\| dt$$

$$L.I. = \int_C x^2 z \, ds$$

$$C: \begin{cases} x = \cos(t) \\ y = 2t \qquad 0 \le t \le \pi \\ z = \sin(t) \end{cases}$$

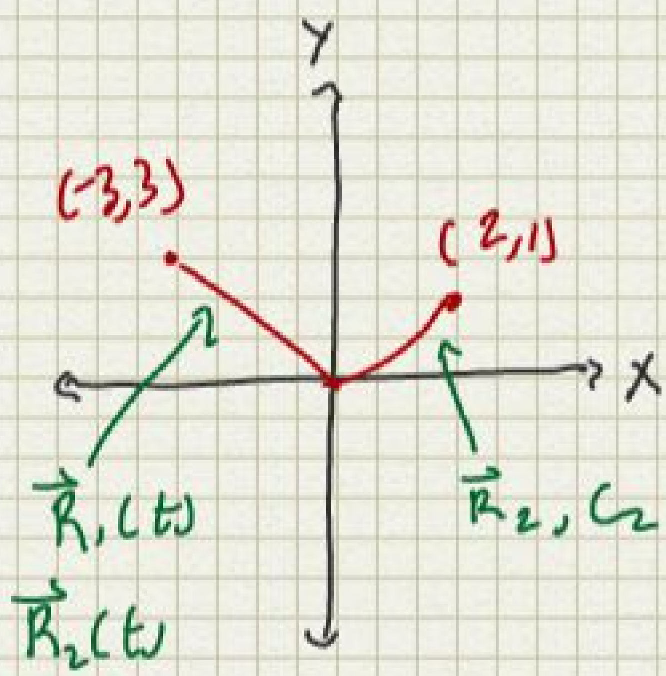
$$\|\vec{R}'\| = \sqrt{(-\sin(t))^2 + (2)^2 + (\cos(t))^2} = \sqrt{5}$$

$$L.I. = \int_0^{\pi} (\cos(t))^2 (\sin(t)) \sqrt{5} \, dt \qquad u = -\cos(t), \quad du = \sin(t)\, dt$$

$$= \sqrt{5} \int u^2 \, dt = \sqrt{5} \left. \frac{u^3}{3} \right| = \sqrt{5} \left. \frac{(-\cos(t))^3}{3} \right|_0^{\pi}$$

$$= \frac{\sqrt{5}}{3}(2) = \frac{2}{3}\sqrt{5}$$

Ex $\int_C xy \, ds$



$C = C_1 \cup C_2$

C_1 line seg $(-3,3) \to (0,0)$

C_2 $16y = x^4$ $(0,0) \to (2,1)$

$C_2: y = \frac{x^4}{16}$

$C_1: y = -x$

$\vec{R}_1(t) = \langle t, -t \rangle \quad -3 \le t \le 0$

$\vec{R}_2(t) = \langle t, \frac{t^4}{16} \rangle \quad 0 \le t \le 2$

$= \langle 2t, \frac{(2t)^4}{16} \rangle = \langle 2t, t^4 \rangle$

$0 \le t \le 1$

$C_2: y = \frac{x^4}{16} \quad (x,y): (0,0) \to (2,1)$

$x(t) = t$

$R(t) = \langle t, \frac{t^4}{16} \rangle \quad 0 \le t \le 2$ b/c $0 \le x \le 2$

$t_1 = \frac{x}{2}$

$\vec{R}(t_1) = \langle 2t, \frac{16t_1^4}{16} \rangle = \langle 2t_1, t_1^4 \rangle$

$0 \le t_1 \le 1$

$$LI = \int_{-3}^{0} (t)(-t)\sqrt{(1)^2+(-1)^2}\,dt + \int_{0}^{1} (2t)\, t^4 \sqrt{(2)^2+(4t^3)^2}\,dt$$

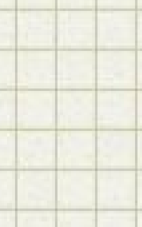
$$= \sqrt{2}\left(-\frac{t^3}{3}\right)\Big|_{-3}^{0} + 2\int_{0}^{1} t^5 \sqrt{4+16t^6}\,dt$$

$u = 4+16t^5$

$du = (16\cdot 6)\, t^5\, dt$

$$= 9\sqrt{2} + \frac{2}{16\cdot 6}\int_{---}^{---} \sqrt{u}\,du \quad 9\sqrt{2} + \frac{1}{48}\left(\frac{2}{3}\right)u^{3/2}\Big|_{---}^{---}$$

$$= 9\sqrt{2} + \frac{1}{72}(4+16t^6)^{3/2}\Big|_0^1 = 9\sqrt{2} + \frac{1}{72}\left(20^{3/2} - 4^{3/2}\right)$$



$W = mg$

$\vec{F}(x,y,z)$

$C: \vec{R}(t) = \langle x(t), y(t), z(t)\rangle \quad t_0 \le t \le t_1$

$$L.I = \int_C (\vec{F}\cdot\vec{T})\, ds \qquad \vec{T} = \frac{\langle x'(t), y'(t), z'(t)\rangle}{\|\vec{R}'(t)\|}$$

$$= \int_{t_0}^{t_1} \vec{F}(x(t), y(t), z(t)) \cdot \frac{x'(t), y'(t), z'(t)}{\cancel{\|\vec{R}'(t)\|}} \left(\cancel{\|\vec{R}'(t)\|}\right) dt$$

$$= \int_{t_0}^{t_1} \vec{F}(\vec{R}(t)) \cdot \vec{R}'(t)\, dt = \int_C \vec{F} \cdot d\vec{R}$$