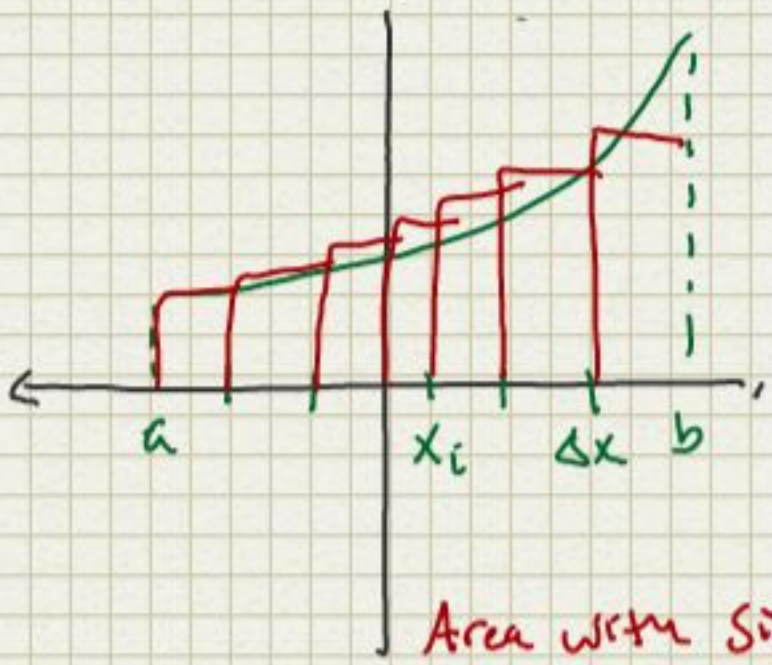
3/4/2014

Multiple Integration



$f(x)$

$\Delta x = \frac{b-a}{m}$

$$\int_a^b f(x)\,dx = \lim_{m \to \pm\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

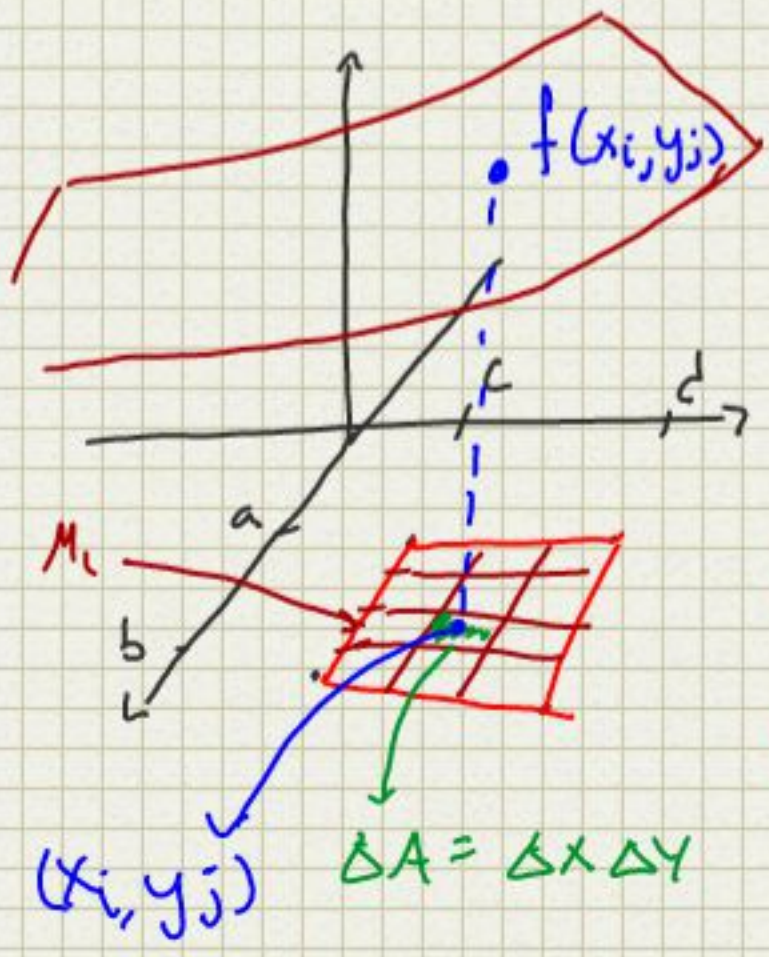
Area with sign of f over the interval (a,b)

if $F(x)$ is antiderivative of f $\quad F'(x) = f(x)$

Linearity Rule

$$\int_a^b c\,f(x) + d\,g(x)\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$\iint_R f(x,y)\,dA = \lim_{(M_1, M_2) \to (\infty, \infty)} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} f(x_i, y_j)\Delta A$$

$M_1 \quad \Delta x = \frac{b-a}{M_1}$

$M_2 \quad \Delta y = \frac{d-c}{M_2}$

$\Delta A = (\Delta x)(\Delta y)$

$\Downarrow$

Volume w/ sign of f over the Rectangular region R

$$\iint_R l\, f(x,y) + m\, g(x,y)\, dA = l \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$R = R_1 + R_2 \quad \iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

Iterated Interval

$f(x,y)$ $\quad R: \begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$

1) $$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dy = \int_a^b F(x,y)\Big|_c^d dx = \int_a^b \overbrace{\left(F(x,d) - F(x,c) \right)}^{g(x)} dx$$

$F_y(x,y) = f(x,y)$

$$= G(x)\Big|_a^b$$

$$= G(b) - G(a)$$

2) $$\int_c^d \int_a^b f(x,y)\, dx \quad dy = \int_c^d H(x,y)\Big|_b^a dy = \int_c^d H(b,y) - H(a,y)\, dy$$

$H_x = f(x,y)$

$$= L(y)\Big|_c^d$$

$L'(y) = l(y)$

$$= L(d) - L(c)$$

Fubini's theorm

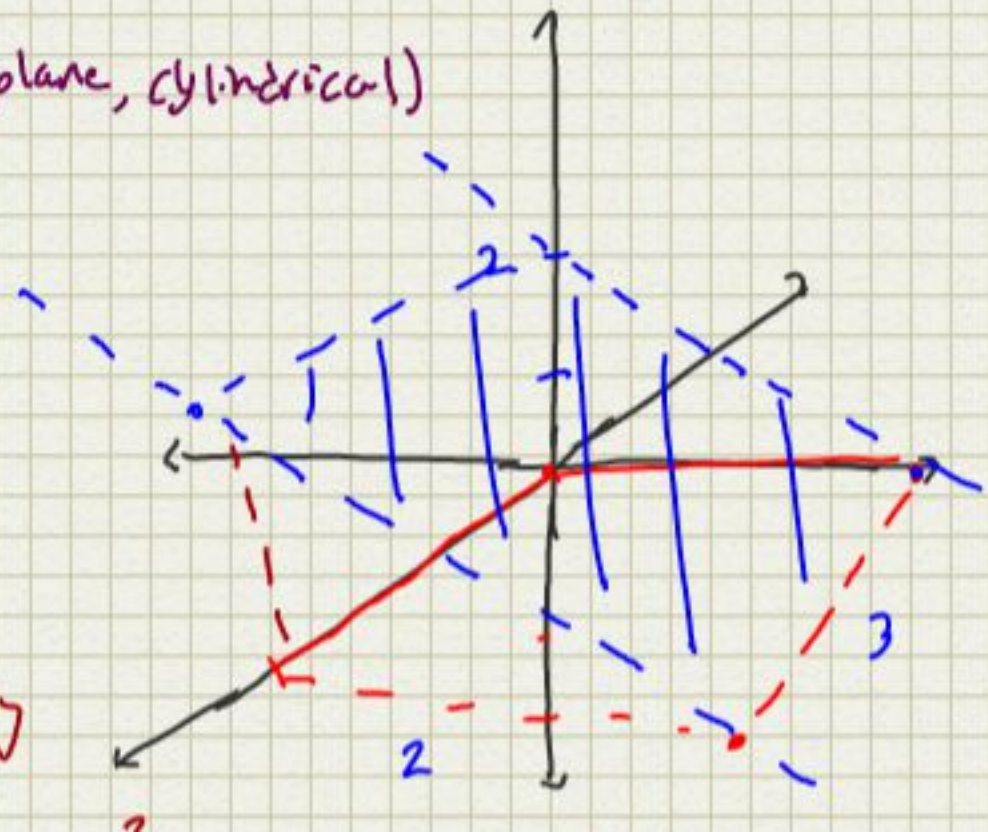
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Ex: $\iint_R (2-y)\,dA$ values $(0,0)$, $(3,0)$, $(3,2)$, $(0,2)$

$f(x,y) = 2-y$ (plane, cylindrical)

$y + z = 2$

Area is 6



$\iint_R (2-y)\,dA$

$= \int_0^3 \left(\int_0^2 (2-y)\,dy \right) dy$

$= \int_0^3 \left(2y - \frac{y^2}{2}\right)\Big|_0^2 dx = \int_0^3 \left(\left(2\cdot 2 - \frac{4}{2}\right) - 0 \right) dx$

$= \int_0^3 2\,dx = 2x\Big|_0^3 = 6 - 0 = 6$

Ex

$$\iint_R x^2y^5\,dA \quad R: \begin{cases} 1 \le x \le 2 \\ 0 \le y \le 1 \end{cases}$$

$$\int_0^1 \int_1^2 (x^2y^5)\,dx\,dy = \int_1^2 \int_0^1 (x^2y^5)\,dy\,dx$$

y is constant

$$\int_0^1 y^5 \int_1^2 x^2\,dx\,dy$$

$$\Downarrow$$

x is constant

$$\int_1^2 x^2\,dx \cdot \int_0^1 y^5\,dy$$

$$\int_1^2 x^2 \int_0^1 y^5\,dy\,dx$$

same

$$\int_1^2 x^2\,dx \cdot \int_0^1 y^5\,dy$$

$$\frac{1}{3}x^3\Big|_1^2 \cdot \frac{1}{6}y^6\Big|_0^1$$

$$\left(\frac{8}{3} - \frac{1}{3}\right) \cdot \left(\frac{1}{6} - 0\right)$$

$$= \frac{7}{18}$$

$$\int_a^b \int_c^d f(x)\,g(y)\,dx\,dy$$

$$= \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

$$= \int_c^d \int_a^b f(x)\,g(y)\,dy\,dx$$

$$\iint_R x\cos(xy)\,dA \qquad R: \begin{cases} 0 \le x \le \pi/2 \\ 0 \le y \le 1 \end{cases}$$

$$\int_0^{\pi/2}\int_0^1 x\cos(xy)\,dy\,dx$$

$$\int_0^1 \left(\int_0^{\pi/2} x\cos(xy)\,dx \right) dy$$

Int by parts

U Substitution

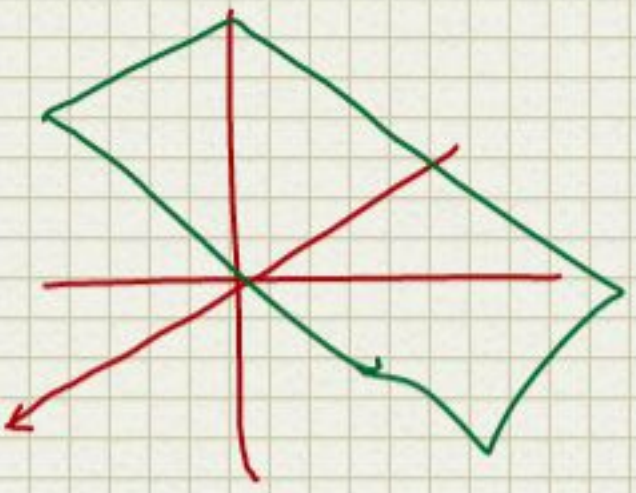
$u = xy$

$du = x\,dy$

$$\int_0^{\pi/2} \sin(xy)\Big|_0^1 = \int_0^{\pi/2} \sin(x) - \sin(0)\,dx = \int_0^{\pi/2} \sin(x)\,dx$$

$$= -\cos(x)\Big|_0^{\pi/2} = -\cos(\pi/2) + \cos(0) = 0 + 1 = 1$$

(x, y, z) $z = 3x + 4y + 3$ closest to $(0, 0, 0)$



$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$z = 3x + 4y + 3$$

min

$$d^2 = f = x^2 + y^2 + z^2$$

$$z = 3x + 4y + 3$$

1) substitution

$$f(x, y, 3x + 4y + 3)$$

$$= x^2 + y^2 + (3x + 4y + 3)^2$$

$$f_x = 0 \quad 2x + 2(3x + 4y + 3)3 \qquad \begin{cases} 10x + 12y + 9 = 0 & \cdot (6) \\ 12x + 17x + 12 = 0 & \cdot (-5) \end{cases}$$

$$f_y = 0 \quad 2y + 2(3x + 4y + 3)4$$

$$0x + 72y - 85y + 54 - 60 = 0$$

$$y = \frac{6}{13}$$

$$10x + 12\left(\frac{6}{13}\right) + 9 = 0$$

$$10x + \frac{-72 + 117}{13} = 0$$

$$x = \frac{-45}{13}$$

<u>La Grange Multiplier</u>

$$\nabla f = \lambda \nabla g \qquad g = 0 = 3x + 4y - z + 3$$

$$2x = \lambda(3) \qquad x = \frac{3}{2}\lambda \qquad 3 \cdot \frac{3}{2}\lambda + 4 \cdot 2\lambda + \frac{\lambda}{2} + 3 = 0$$

$$2y = \lambda(4) \qquad y = 2\lambda \qquad \lambda = ?$$

$$2z = \lambda(-1) \qquad z = -\frac{\lambda}{2}$$

$$3x + 4y - z + 3 = 0$$

3/4/2014

Test

11) Limits
Implicit Diff
Chain Rule
D. Der
Gradient
Tangent Planes
Rel Extrema & Classification
ABS Extrema

10.1 $\vec{N}$, $\vec{T}$ & κ

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7 $f(x,y) = x^2 - 2y - y^2$ $\quad g(x) = x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x = \lambda 2x \\ -2 - 2y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} x = 0 & (1) \\ y = \pm 1 & (3) \\ y = 1 & -2 - 2 = 2\lambda \\ y = -1 & -2 + 2 = -2\lambda \end{cases}$$

$$\begin{cases} \lambda = 1 & (1) \\ -2 - 2y = 2y \\ y = -1/2 & (2) \\ x^2 + \frac{1}{4} = 1 & x = \frac{\sqrt{3}}{2} \end{cases}$$

$f(0, \pm 1)$

$f\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$z = x + 2y + 4$? closest to origin

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$x + 2y - z + 4 = 0$

$g = x + 2y - z + 4 = 0$

$f = (d^2) = x^2 + y^2 + z^2$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda 2 \\ 2z = -\lambda \\ x + 2y - z + 4 = 0 \end{cases}$$

$x = \frac{\lambda}{2}$

$y = \lambda$

$z = -\frac{\lambda}{2}$

$\frac{\lambda}{2} + 2\lambda + \frac{\lambda}{2} + 4 = 0$

$3\lambda = 4 \quad \lambda = -\frac{4}{3}$

$x = \frac{-\frac{4}{3}}{2} = -2/3$

$y = -\frac{4}{3}$

$z = 2/3$

$f(x,y) = xy(1 - 10x - y)$ 4 critical points

$$\begin{cases} f_x = 0 = y(1 - 20x - y) \\ f_y = 0 = x(1 - 10x - 2y) \end{cases}$$

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \quad \begin{cases} y = 0 \\ 1 - 20x - y \end{cases} \quad \begin{cases} 1 - 20x - y = 0 \\ x = 0 \end{cases} \quad \begin{cases} 1 - 20x - y = 0 \quad (-2) \\ 1 - 10x - 2y = 0 \end{cases}$$

$P_1(0,0) \quad P_2(\frac{1}{10}, 0) \quad P_3(0,1) \quad P_4(\frac{1}{30}, \frac{1}{3})$

<u>Classify</u>

$D = f_{xx} f_{yy} - f_{xy}^2$

$D = 40xy - (1 - 20x - 2y)^2$

$D_1(0,0) = 0 - 1^2 = -1 < 0$ SP

$D_2(\frac{1}{10}, 0) = -1 < 0$ SP

$D_3(0,1) = -1 < 0$ SP

$D_4(\frac{1}{30}, \frac{1}{3})$ $\frac{1}{3} > 0$ R max

HW 5

1 a)

$\sin(x+y) + \tan(x+y) = 1$ $P(\frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4})$

Tangent Plane at P

Implizit so $F(x,y,z) = \sin(x+y) + \tan(x+y) = 1$

$\hat{N} = \frac{\nabla F(P)}{\|\nabla F(P)\|}$ $\hat{N} = \frac{\langle \cos(x+y), \cos(x+y) + \sec^2(x+y), \sec^2(x+y) \rangle}{\cdot}$

$\vec{N}(P) = \langle 0, 1, 1 \rangle$

$\hat{N}(P) = \frac{\langle 0, 1, 1 \rangle}{\sqrt{2}}$

$Ax + By + Cz = Ax_0 + By_0 + Cz_0$

$0 + y + z = 0 + \cancel{\frac{\pi}{4} - \frac{\pi}{4}}$ $y + z = 0$

9

$w = \ln(x + 2y - z^2)$

$x = 2t - 1$

$y = \frac{1}{t}$

$z = \sqrt{t}$

$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$

$w_x = \frac{1}{x + 2y - z^2}$ $\quad w_y = \text{''}\ 2 \quad w_z = \text{''}\ (-2z)$

$\frac{dx}{dt} = 2 \quad \frac{dy}{dt} = -\frac{1}{t^2} \quad \frac{dz}{dt} = \frac{1}{2\sqrt{t}}$

$\frac{dw}{dt} = \frac{1}{x + 2y - z^2}\left(2 - \frac{2}{t^2} - \frac{\quad}{2\sqrt{t}}\right.$

$\lim_{(x,y) \to (0,0)} \frac{2x^2}{2x^2 + 3y^2}$

$x = 0 \Rightarrow \lim_{y \to 0} \frac{0}{0 + 3y^2} = 0$

$y =$

K for function $\langle -5\sin(t), -5\sin(t), -2\cos(t) \rangle$

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|} = \frac{\|\vec{R}' \times \vec{R}''\|}{\|R'(t)\|^3}$$

$$R'(t) = \frac{d}{dt}(f) = \langle -5\cos(t), -5\cos(t), 2\sin(t) \rangle$$

$$\|R'(t)\| = \sqrt{50\cos^2(t) + 4\sin^2(t)} = \sqrt{4 + 46\cos^2(t)}$$

$$\vec{R}''(t) = \langle 5\sin(t), 5\sin(t), 2\cos(t) \rangle$$

$$\vec{R}' \times \vec{R}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5\cos(t) & -5\cos(t) & 2\sin(t) \\ 5\sin(t) & 5\sin(t) & 2\cos(t) \end{vmatrix}$$

$$= (-10\cos^2(t) - 10\sin^2(t))\hat{i} - (-10\cos^2(t) - 10\sin^2(t))\hat{j}$$

$$+ (-25\sin\cos + 25\cos\sin) = \langle -10, 10, \emptyset \rangle$$