10.4 2/4/2014

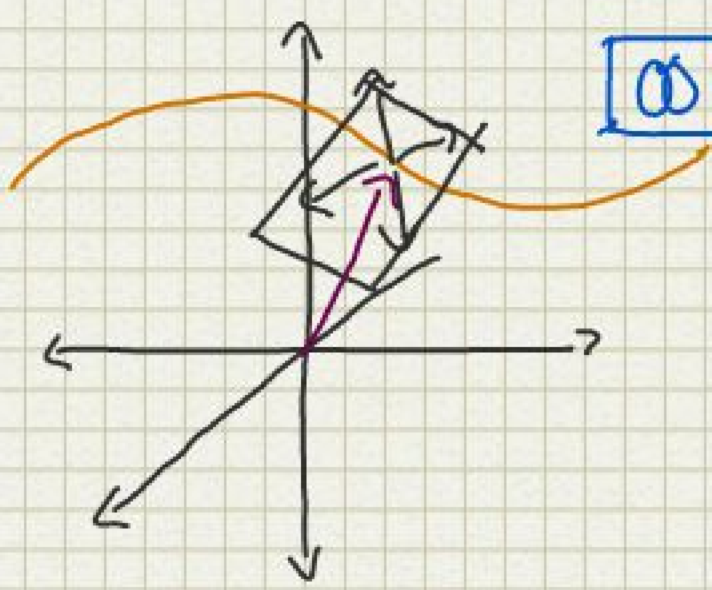
Test 9.1 → 9.6
10.1, 10.2

Unit tangent Vector

$\vec{R}(t)$ is smooth

$\vec{T} = \vec{R}'(t) \qquad \hat{T} = \frac{\vec{R}'(t)}{(\vec{R}'(t))}$

$R'(t)$ cannot be $\emptyset$ b/c $\vec{R}(t)$ is smooth.



∞ unit normal vector

Principle unit Normal Vector

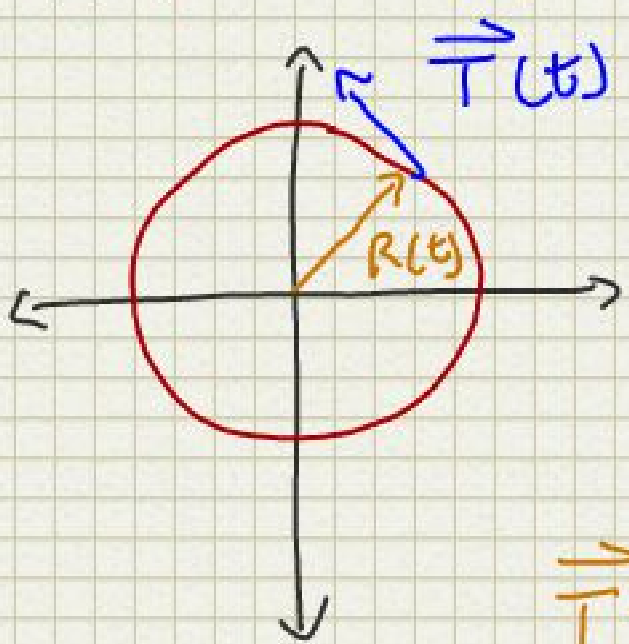
$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$$

If $\|R'\|$ is constant, $\vec{R}' \perp \vec{R}$

so, if $\|\vec{T}\|$ is constant, $\vec{T}' \perp \vec{T}$

So $\vec{N} \perp \vec{T}$

$\vec{R}(t) = \langle \cos(t), \sin(t), \emptyset \rangle$



points in the direction the line is bending. So $N(t)$ is pointing into center of circle.

$N(t) = -R(t)$

$$\vec{T} = \frac{\vec{R}'}{\|\vec{R}'\|} = \frac{\langle -\sin(t), \cos(t), \emptyset \rangle}{\sqrt{\sin^2(t) + \cos^2(t)}}$$

$$\vec{N} = \frac{\langle -\cos(t), -\sin(t), \emptyset \rangle}{\sqrt{\cos^2(t) + \sin^2(t)}} = -\vec{R}(t)$$

$\vec{R}(t) = \langle 2t, 3\cos(2t), 3\sin(2t) \rangle$

*helix spiriling towards positive x-axis

$\vec{T} = ? = R'(t) / \|R'(t)\|$

$\vec{N} = ? = \vec{T}' / \|\vec{T}\|$

$$\vec{T} = \frac{\langle 2, -6\sin(2t), 6\cos(2t) \rangle}{\sqrt{2^2 + 36\sin^2(2t) + 36\cos(2t)}}$$

$$= \frac{\langle 2, -6\sin(2t), 6\cos(2t) \rangle}{2\sqrt{10}}$$

$$= \frac{\langle 1, -3\sin(2t), 3\sin(2t) \rangle}{\sqrt{10}}$$

$$\vec{N} = \frac{\frac{1}{\sqrt{10}} \langle 0, -6\cos(2t), -6\sin(2t) \rangle}{\frac{1}{\sqrt{10}} \sqrt{36\cos^2(2t) + 36\sin^2(2t)}}$$

$$= \langle 0, -\cos(2t), -\sin(2t) \rangle$$

Ex: Final Test-Like Problem

$R(t) = \langle e^t \sin(t), e^t \cos(t), 0 \rangle$

$\vec{T}(t) = ? = \frac{R'(t)}{\|R'(t)\|}$

$\vec{N}(t) = ?$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\vec{T}(t) = \frac{\langle e^t\cos(t) + e^t\sin(t), -e^t\sin(t) + e^t\cos(t), 0 \rangle}{\|R'(t)\|}$$

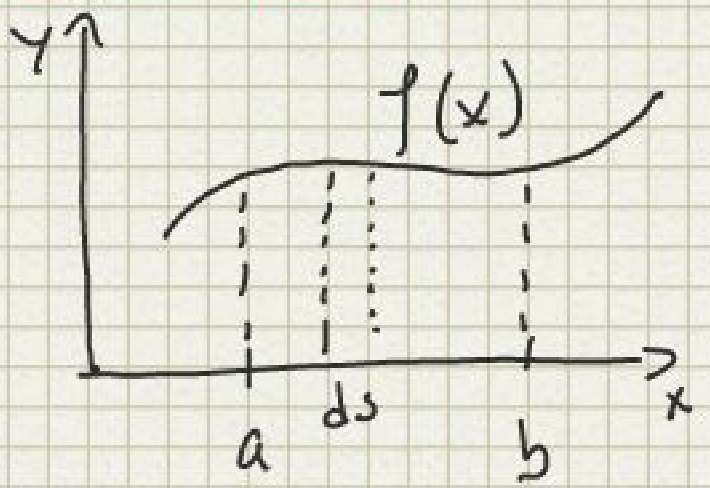
$$= \frac{e^t \langle \cos(t) + \sin(t), -\sin(t) + \cos(t) \rangle}{\frac{\sqrt{(e^t)^2 (\cos(t) + \sin(t))^2 (\cos(t) - \sin(t))^2}}{\sqrt{\cos^2(t) + \cancel{2\sin(t)\cos(t)} + \sin^2(t) + \cos^2(t) - \cancel{2\sin(t)\cos(t)} + \sin^2(t)}}}$$

$$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle \cos(t) + \sin(t), \cos(t) - \sin(t), 0 \rangle$$

$$\vec{N}(t) = \frac{\frac{1}{\sqrt{2}} \langle -\sin(t) + \cos(t), -\sin(t) - \cos(t), 0 \rangle}{\frac{1}{\sqrt{2}} \sqrt{(-\sin(t) + \cos(t))^2 + (-\sin(t) - \cos(t))^2}} \rightarrow \frac{1}{\sqrt{2}}$$

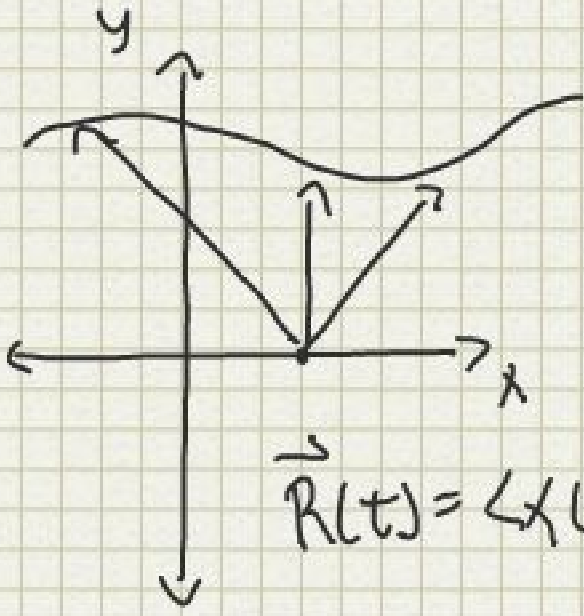
$$= \frac{1}{\sqrt{2}} \langle -\sin(t) + \cos(t), -\sin(t) - \cos(t), 0 \rangle$$

Arc Length



$$S=\int_a^b \sqrt{1+(f')^2}\,dx$$

$$ds=\sqrt{1+(f')^2}\,dx$$



$$\vec{R}(t)=\langle x(t), y(t)\rangle$$

$$f=y(x) \qquad f'=\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{y'(t)}{x'(t)}$$

$$S=\int_{t_0}^{t_1}\sqrt{1+\frac{(y'(t))^2}{(x'(t))^2}}\,dx=\int_{t_0}^{t_1}\frac{\sqrt{(x'(t))^2+(y'(t))^2}}{(x'(t))}\,dx$$

$$=\int_{t_0}^{t_1}\frac{\sqrt{(x'(t))^2+(y'(t))^2}\,dx\,dt}{\frac{dx}{dt}}=\int_{t_0}^{t_1}\sqrt{(x')^2+(y')^2}\,dt$$

$$S = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$ds = \|\vec{R}'\|$$

$$\vec{R}' = \langle 0, 0, 0 \rangle$$

$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ smooth

$t_1 = 2t$

$\vec{R}(t_1)$

$\boxed{\text{the arc length is Constant of the parametrization.}}$

$$S = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$= \int_{u_0}^{u_1} \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\, du = \text{fixed}$$

$$S = \int_{t_0}^{t_1} \|\vec{R}'\|\, dt$$

$$= \int_A^B ds$$

$$s(t) = \int_{s_0}^{s} \|\vec{R}'(t)\|\, dt \qquad \frac{ds}{dt} = \|\vec{R}'(t)\|$$

Ex

$\vec{R}(t) = \langle 3\cos(4t), 3\sin(4t) \rangle \qquad t_0 = \varnothing$

$\vec{R}(s) = ?\ \vec{R}(t(s)) \qquad R'(t) = \langle -12\sin(4t), 12\cos(4t) \rangle$

$s(t) = t(s)$

$$s = \int_{\varnothing}^{t} \sqrt{144\sin^2(4t) + 144\cos^2(4t)}\, dt$$

$$= \int_{\varnothing}^{t} 12\, dt = \underline{12t}$$

$s(t) \Longleftrightarrow t(s)$

$s = 12t \qquad t = \frac{s}{12}$

$\vec{R}(s) = \langle 3\cos(4 \cdot \frac{s}{12}) + 3\sin(4 \cdot \frac{s}{12}) \rangle$

$= \langle 3\cos(\frac{s}{3}), 3\sin(\frac{s}{3}) \rangle$

$\| R'(s) \| = 1$

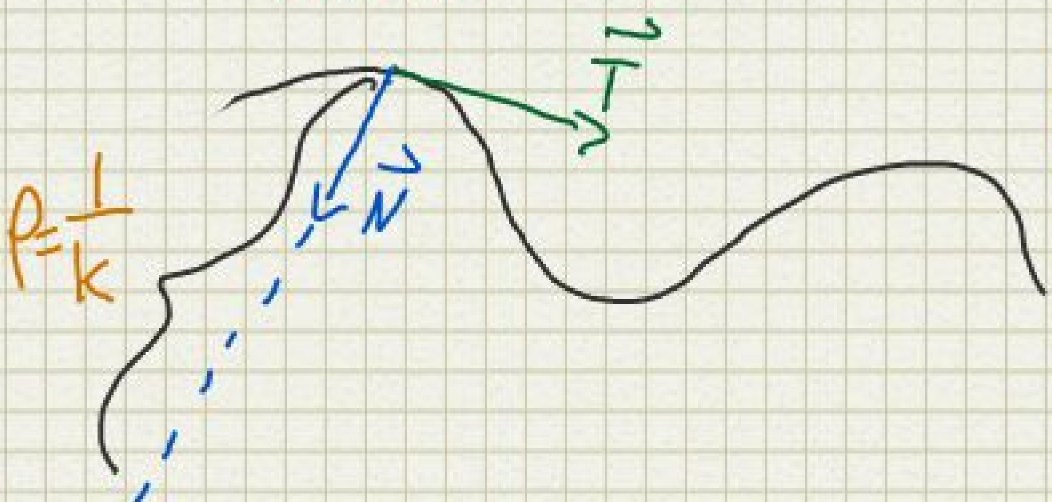
$1 = \frac{ds}{ds} = \| \vec{R}'(s) \|$

Curvature

- κ - Kappa
- the curvature of a smooth function $\vec{R}(t)$

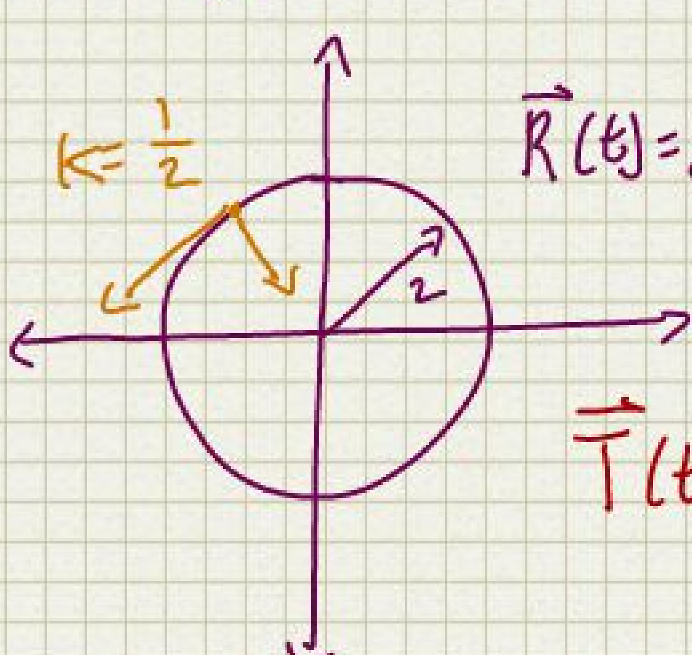
$$\kappa \begin{cases} \|\vec{T}'(s)\| \\ \dfrac{\|\vec{T}'(t)\|}{\|\vec{R}'(t)\|} \\ \dfrac{\vec{R}' \times \vec{R}''}{\|\vec{R}'\|^3} \end{cases}$$

= arc length parametrization $\dfrac{\|T'(s)\|}{\cancel{\|\vec{R}'(s)\|}}$ $\to 1$



$\rho = \frac{1}{\kappa}$ is the radius of the circle that best approximates the curve

Osculator Circle



$\vec{R}(t) = \langle 2\cos(2t), 2\sin(2t) \rangle$

$\|\vec{R}'(t)\| = 4$

Curvature is inverse of radius

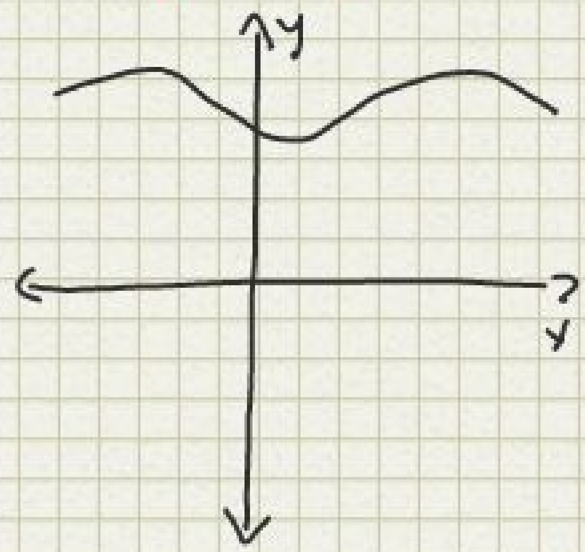
$$\vec{T}(t) = \frac{(4(-\sin(2t)), 4\cos(2t))}{4}$$

$$= \langle -\sin(2t), \cos(2t) \rangle$$

$$\vec{T}'(t) = 2\langle -\cos(2t), -\sin(2t) \rangle$$

$$\|\vec{T}'\| = 2 \qquad \kappa = \frac{2}{4} = \frac{1}{2}$$

$y(x)$, function in $x-y$ plane



$$k = \frac{|y''(x)|}{(1+(y'(x))^2)^{3/2}}$$