Astral diffusion as a limit process for symmetric random walk in a high contrast periodic medium

Elena Zhizhina

Institute for Information Transmission Problems
Russian Academy of Sciences, Moscow

Joint work with Andrey Piatnitski

26 March 2021

A. Piatnitski; E. Zhizhina, Scaling limit of symmetric random walk in high-contrast periodic environment. *J. Stat. Phys.*, **169** (3) (2017), 595–613.

Similar results for parabolic parabolic equations in high contrast periodic environments

Piatnitski, A.; Pirogov, S.; Zhizhina, E. Limit behaviour of diffusion in high-contrast periodic media and related Markov semigroups. *Applicable Analysis*, **98**(1-2) (2019), 217–231.

A. Piatnitski; E. Zhizhina, Scaling limit of symmetric random walk in high-contrast periodic environment. *J. Stat. Phys.*, **169** (3) (2017), 595–613.

Similar results for parabolic parabolic equations in high contrast periodic environments

Piatnitski, A.; Pirogov, S.; Zhizhina, E. Limit behaviour of diffusion in high-contrast periodic media and related Markov semigroups. *Applicable Analysis*, **98**(1-2) (2019), 217–231.

Motivation: double porosity model

Homogenization problem for parabolic equation with high-contrast periodic coefficients. Existing results.

T. Arbogast, J. Douglas, and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.*, **21**(1990), pp. 823–836.

The authors derived the limit macroscopic model, showed the memory effect in the limit equation and proved homogenization result. Their method is based on classical homogenization arguments.

Further contributions:

G. Allaire, A. Bourgeat, V. Zhikov, B. Amaziane, A. Mikelic, A. Piatnitski, ...

Motivation: double porosity model

Homogenization problem for parabolic equation with high-contrast periodic coefficients. Existing results.

T. Arbogast, J. Douglas, and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.*, **21**(1990), pp. 823–836.

The authors derived the limit macroscopic model, showed the memory effect in the limit equation and proved homogenization result. Their method is based on classical homogenization arguments.

Further contributions:

G. Allaire, A. Bourgeat, V. Zhikov, B. Amaziane, A. Mikelic, A. Piatnitski, ...

Motivation: double porosity model

Homogenization problem for parabolic equation with high-contrast periodic coefficients. Existing results.

T. Arbogast, J. Douglas, and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.*, **21**(1990), pp. 823–836.

The authors derived the limit macroscopic model, showed the memory effect in the limit equation and proved homogenization result. Their method is based on classical homogenization arguments.

Further contributions:

G. Allaire, A. Bourgeat, V. Zhikov, B. Amaziane, A. Mikelic, A. Piatnitski, ...

Classical double porosity model

Classical double porosity model (micro-scale):

$$\begin{cases} \partial_t u^\varepsilon = \operatorname{div}\big(a^\varepsilon(x)\nabla u^\varepsilon\big) & x \in \mathbb{R}^d, \quad t > 0 \\ u^\varepsilon|_{t=0} = u_0(x), \end{cases}$$

with

$$a^\varepsilon(x) = \begin{cases} 1, & \text{if } x \in F^\varepsilon \\ \varepsilon^2, & \text{if } x \in M^\varepsilon; \end{cases}$$

here $M^\varepsilon = \varepsilon M$, and M is the union of periodically situated bounded Lipschitz domains such that the distance between any two such domains is bounded from below by a positive constant; $F^\varepsilon = \mathbb{R}^d \setminus M^\varepsilon$.

Classical double porosity model

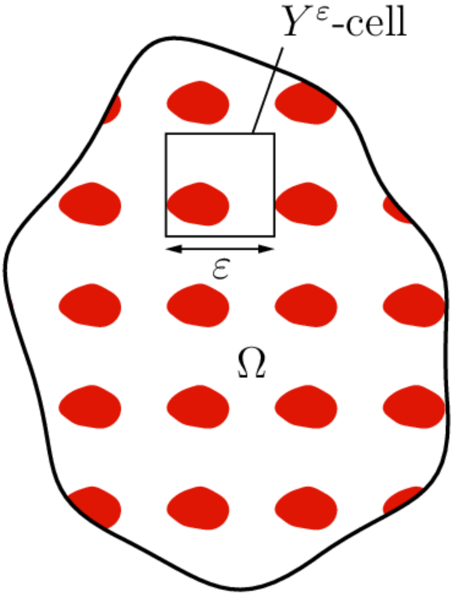
Classical double porosity model (micro-scale):

$$\begin{cases} \partial_t u^\varepsilon = \mathrm{div}\big(a^\varepsilon(x)\nabla u^\varepsilon\big) & x \in \mathbb{R}^d, \quad t > 0 \\ u^\varepsilon|_{t=0} = u_0(x), & \end{cases}$$

with

$$a^\varepsilon(x) = \begin{cases} 1, & \text{if } x \in F^\varepsilon \\ \varepsilon^2, & \text{if } x \in M^\varepsilon; \end{cases}$$

here $M^\varepsilon = \varepsilon M$, and M is the union of periodically situated bounded Lipschitz domains such that the distance between any two such domains is bounded from below by a positive constant; $F^\varepsilon = \mathbb{R}^d \setminus M^\varepsilon$.


Y^ε-cell
ε
Ω

Memory effect

Under the diffusive scaling $x \to \varepsilon x$, $t \to \varepsilon^2 t$ the limit evolution of $u(t,x)$, as $\varepsilon \to 0$, is not Markov:

$$\partial_t u(x,t) = \operatorname{div}\big(a^{\text{eff}} \nabla u(x,t)\big) + \int_0^t D(t-s)u(s,x)ds$$

with an exponentially decaying function $D(s)$:

$$D(s) \le C \exp(-\gamma s), \qquad \text{for some } \gamma > 0.$$

.

Extended Markov process

Question: Does there exist a Markov process behind this evolution?

Answer: Yes, it does.

It is a Markov process on an extended state space = "spatial" component related to the fast movement + "astral" component related to the slow movement.

Extended Markov process

Question: Does there exist a Markov process behind this evolution?

Answer: Yes, it does.

It is a Markov process on an extended state space = "spatial" component related to the fast movement + "astral" component related to the slow movement.

General assumptions on transition probabilities

We consider a **symmetric** random walk $\widehat{X}(n)$ on $\mathbb{Z}^d,\ d \geq 1$, with transition probabilities $p(x,y) = \Pr(x \to y)$, $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$:

$$p(x,y) = p(y,x), \quad (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \qquad \sum_{y \in \mathbb{Z}^d} p(x,y) = 1 \ \ \forall x \in \mathbb{Z}^d.$$

We assume that the random walk satisfies the following properties:

- **Periodicity**. The functions $p(x, x+\xi)$ are periodic in x with a period Y for all $\xi \in \mathbb{Z}^d$. In what follows we identify the period Y with the corresponding d-dimensional discrete torus $\mathbb{T}^d$.
- **Finite range of interactions**. There exists $c_1 > 0$ such that

$$p(x, x+\xi) = 0, \quad \text{if } |\xi| > c_1.$$

- **Irreducibility.** The random walk is irreducible in $\mathbb{Z}^d$.

We denote the transition matrix of the random walk by

$$P = \{p(x,y),\ x,y \in \mathbb{Z}^d\}.$$

General assumptions on transition probabilities

We consider a **symmetric** random walk $\widehat{X}(n)$ on $\mathbb{Z}^d$, $d \geq 1$, with transition probabilities $p(x,y) = \Pr(x \to y)$, $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$:

$$p(x,y) = p(y,x), \quad (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \qquad \sum_{y \in \mathbb{Z}^d} p(x,y) = 1 \;\; \forall x \in \mathbb{Z}^d.$$

We assume that the random walk satisfies the following properties:

- **Periodicity**. The functions $p(x, x+\xi)$ are periodic in x with a period Y for all $\xi \in \mathbb{Z}^d$. In what follows we identify the period Y with the corresponding d-dimensional discrete torus $\mathbb{T}^d$.
- **Finite range of interactions**. There exists $c_1 > 0$ such that

$$p(x, x+\xi) = 0, \quad \text{if } |\xi| > c_1.$$

- **Irreducibility.** The random walk is irreducible in $\mathbb{Z}^d$.

We denote the transition matrix of the random walk by

$$P = \{p(x,y), \; x, y \in \mathbb{Z}^d\}.$$

General assumptions on transition probabilities

We consider a **symmetric** random walk $\widehat{X}(n)$ on $\mathbb{Z}^d$, $d \geq 1$, with transition probabilities $p(x,y) = \Pr(x \to y)$, $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$:

$$p(x,y) = p(y,x), \quad (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \qquad \sum_{y \in \mathbb{Z}^d} p(x,y) = 1 \ \ \forall x \in \mathbb{Z}^d.$$

We assume that the random walk satisfies the following properties:

- **Periodicity**. The functions $p(x, x+\xi)$ are periodic in x with a period Y for all $\xi \in \mathbb{Z}^d$. In what follows we identify the period Y with the corresponding d-dimensional discrete torus $\mathbb{T}^d$.
- **Finite range of interactions**. There exists $c_1 > 0$ such that

$$p(x, x+\xi) = 0, \quad \text{if } |\xi| > c_1.$$

- **Irreducibility.** The random walk is irreducible in $\mathbb{Z}^d$.

We denote the transition matrix of the random walk by

$$P = \{p(x,y),\ x, y \in \mathbb{Z}^d\}.$$

General assumptions on transition probabilities

We consider a **symmetric** random walk $\widehat{X}(n)$ on $\mathbb{Z}^d,\ d \geq 1$, with transition probabilities $p(x,y) = \Pr(x \to y)$, $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$:

$$p(x,y) = p(y,x), \quad (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d; \qquad \sum_{y \in \mathbb{Z}^d} p(x,y) = 1 \ \ \forall x \in \mathbb{Z}^d.$$

We assume that the random walk satisfies the following properties:

- **Periodicity**. The functions $p(x, x+\xi)$ are periodic in x with a period Y for all $\xi \in \mathbb{Z}^d$. In what follows we identify the period Y with the corresponding d-dimensional discrete torus $\mathbb{T}^d$.
- **Finite range of interactions**. There exists $c_1 > 0$ such that

$$p(x, x+\xi) = 0, \quad \text{if } |\xi| > c_1.$$

- **Irreducibility.** The random walk is irreducible in $\mathbb{Z}^d$.

We denote the transition matrix of the random walk by

$$P = \{p(x,y),\ x, y \in \mathbb{Z}^d\}.$$

Structure of the periodicity cell $Y = A \cup B$

The periodicity cell Y is divided into two sets

$$Y = A \cup B; \quad A,\, B \neq \emptyset,\ A \cap B = \emptyset.$$

Let $A^\sharp$, $B^\sharp$ be the periodic extension of A and B.

Then

$$\mathbb{Z}^d = A^\sharp \cup B^\sharp.$$

We assume that B is a connected set and its periodic extension $B^\sharp$ is unbounded and connected.

Structure of the periodicity cell $Y = A \cup B$

The periodicity cell Y is divided into two sets

$$Y = A \cup B; \quad A,\, B \neq \emptyset,\; A \cap B = \emptyset.$$

Let $A^\sharp$, $B^\sharp$ be the periodic extension of A and B.

Then

$$\mathbb{Z}^d = A^\sharp \cup B^\sharp.$$

We assume that B is a connected set and its periodic extension $B^\sharp$ is unbounded and connected.

Structure of the periodicity cell $Y = A \cup B$

The periodicity cell Y is divided into two sets

$$Y = A \cup B; \quad A,\, B \neq \emptyset,\ A \cap B = \emptyset.$$

Let $A^\sharp$, $B^\sharp$ be the periodic extension of A and B.

Then

$$\mathbb{Z}^d = A^\sharp \cup B^\sharp.$$

We assume that B is a connected set and its periodic extension $B^\sharp$ is unbounded and connected.

Transition probabilities for random walk in a high-contrast periodic environment

Let $p^{(\varepsilon)}(x, y)$ be a family of transition probabilities that depend on a small parameter $\varepsilon > 0$ and satisfy for each $\varepsilon > 0$ the properties formulated above.

We suppose that the transition matrix $P^{(\varepsilon)}$ is a small perturbation of a fixed transition matrix P^0:

$$P^{(\varepsilon)} = P^0 + \varepsilon^2 V.$$

These transition probabilities represent the so-called high-contrast periodic structure of the environment.

Transition probabilities for random walk in a high-contrast periodic environment

Let $p^{(\varepsilon)}(x, y)$ be a family of transition probabilities that depend on a small parameter $\varepsilon > 0$ and satisfy for each $\varepsilon > 0$ the properties formulated above.

We suppose that the transition matrix $P^{(\varepsilon)}$ is a small perturbation of a fixed transition matrix P^0:

$$P^{(\varepsilon)} = P^0 + \varepsilon^2 V.$$

These transition probabilities represent the so-called high-contrast periodic structure of the environment.

Transition probabilities for random walk in a high-contrast periodic environment

Let $p^{(\varepsilon)}(x, y)$ be a family of transition probabilities that depend on a small parameter $\varepsilon > 0$ and satisfy for each $\varepsilon > 0$ the properties formulated above.

We suppose that the transition matrix $P^{(\varepsilon)}$ is a small perturbation of a fixed transition matrix P^0:

$$P^{(\varepsilon)} = P^0 + \varepsilon^2 V.$$

These transition probabilities represent the so-called high-contrast periodic structure of the environment.

Definition of fast and slow components P_0 and V

We impose the following conditions on P^0 and V:

- P^0 is a transition matrix of a SRW on $\mathbb{Z}^d$;
- $p_0(x,x) = 1$, if $x \in A^\sharp$;
  (all states in $A^\sharp$ are absorbing states for P^0)
- $p_0(x,y) = 0$, if $x, y \in A^\sharp,\ x \neq y$;
- $p_0(x,y) = 0$, if $x \in B^\sharp,\ y \in A^\sharp$;
- P^0 is irreducible on $B^\sharp$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x,y) = 0 \quad \forall x \in \mathbb{Z}^d.$$

Definition of fast and slow components P_0 and V

We impose the following conditions on P^0 and V:

- P^0 is a transition matrix of a SRW on $\mathbb{Z}^d$;
- $p_0(x,x) = 1$, if $x \in A^\sharp$;
  (all states in $A^\sharp$ are absorbing states for P^0)
- $p_0(x,y) = 0$, if $x, y \in A^\sharp,\ x \neq y$;
- $p_0(x,y) = 0$, if $x \in B^\sharp,\ y \in A^\sharp$;
- P^0 is irreducible on $B^\sharp$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x,y) = 0 \quad \forall x \in \mathbb{Z}^d.$$

Definition of fast and slow components P_0 and V

We impose the following conditions on P^0 and V:

- P^0 is a transition matrix of a SRW on $\mathbb{Z}^d$;
- $p_0(x, x) = 1$, if $x \in A^\sharp$;
  (all states in $A^\sharp$ are absorbing states for P^0)
- $p_0(x, y) = 0$, if $x, y \in A^\sharp,\ x \neq y$;
- $p_0(x, y) = 0$, if $x \in B^\sharp,\ y \in A^\sharp$;
- P^0 is irreducible on $B^\sharp$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x, y) = 0 \quad \forall x \in \mathbb{Z}^d.$$

Definition of fast and slow components P_0 and V

We impose the following conditions on P^0 and V:

- P^0 is a transition matrix of a SRW on $\mathbb{Z}^d$;
- $p_0(x,x) = 1$, if $x \in A^\sharp$;
  (all states in $A^\sharp$ are absorbing states for P^0)
- $p_0(x,y) = 0$, if $x, y \in A^\sharp,\ x \neq y$;
- $p_0(x,y) = 0$, if $x \in B^\sharp,\ y \in A^\sharp$;
- P^0 is irreducible on $B^\sharp$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x,y) = 0 \quad \forall x \in \mathbb{Z}^d.$$

Definition of fast and slow components P_0 and V

We impose the following conditions on P^0 and V:

- P^0 is a transition matrix of a SRW on $\mathbb{Z}^d$;
- $p_0(x,x) = 1$, if $x \in A^\sharp$;
  (all states in $A^\sharp$ are absorbing states for P^0)
- $p_0(x,y) = 0$, if $x, y \in A^\sharp,\ x \neq y$;
- $p_0(x,y) = 0$, if $x \in B^\sharp,\ y \in A^\sharp$;
- P^0 is irreducible on $B^\sharp$;
- the elements of matrix V satisfy the relation

$$\sum_{y \in \mathbb{Z}^d} v(x,y) = 0 \quad \forall x \in \mathbb{Z}^d.$$

ε-random walk: $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$

Under these conditions, for the transition matrix $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$ has the following properties:

- $p(x, y) \asymp 1$, when $x, y \in B^\sharp$ (rapid movement);
- $p(x, x) = 1 + O(\varepsilon^2)$, when $x \in A^\sharp$ (slow movement);
- $p(x, y) \asymp \varepsilon^2$, when $x, y \in A^\sharp,\ x \neq y$ (slow movement);
- $p(x, y) \asymp \varepsilon^2$, when $x \in B^\sharp,\ y \in A^\sharp$ (rare exchange between $A^\sharp$ and $B^\sharp$).

The above choice of the transition probabilities reflects a significant slowdown of the random walk in the slow component $A^\sharp$:

$$B^\sharp = \operatorname{supp}\{\text{fast RW}\}, \quad A^\sharp = \operatorname{supp}\{\text{slow RW}\}.$$

ε-random walk: $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$

Under these conditions, for the transition matrix $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$ has the following properties:

- $p(x,y) \asymp 1$, when $x, y \in B^\sharp$ (rapid movement);
- $p(x,x) = 1 + O(\varepsilon^2)$, when $x \in A^\sharp$ (slow movement);
- $p(x,y) \asymp \varepsilon^2$, when $x, y \in A^\sharp,\ x \neq y$ (slow movement);
- $p(x,y) \asymp \varepsilon^2$, when $x \in B^\sharp,\ y \in A^\sharp$ (rare exchange between $A^\sharp$ and $B^\sharp$).

The above choice of the transition probabilities reflects a significant slowdown of the random walk in the slow component $A^\sharp$:

$$B^\sharp = \mathrm{supp}\,\{\text{fast RW}\}\,, \quad A^\sharp = \mathrm{supp}\,\{\text{slow RW}\}\,.$$

ε-random walk: $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$

Under these conditions, for the transition matrix $P^{(\varepsilon)} = P^0 + \varepsilon^2 V$ has the following properties:

- $p(x, y) \asymp 1$, when $x, y \in B^\sharp$ (rapid movement);
- $p(x, x) = 1 + O(\varepsilon^2)$, when $x \in A^\sharp$ (slow movement);
- $p(x, y) \asymp \varepsilon^2$, when $x, y \in A^\sharp,\ x \neq y$ (slow movement);
- $p(x, y) \asymp \varepsilon^2$, when $x \in B^\sharp,\ y \in A^\sharp$ (rare exchange between $A^\sharp$ and $B^\sharp$).

The above choice of the transition probabilities reflects a significant slowdown of the random walk in the slow component $A^\sharp$:

$$B^\sharp = \operatorname{supp}\,\{\text{fast RW}\}\,, \quad A^\sharp = \operatorname{supp}\,\{\text{slow RW}\}\,.$$

Rescaled process

Let $\varepsilon\mathbb{Z}^d = \{x : \frac{x}{\varepsilon} \in \mathbb{Z}^d\}$ be a compression of the lattice $\mathbb{Z}^d$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon A^\sharp \cup \varepsilon B^\sharp,$$

and we define **the rescaled random walk**

$$\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}(\left[\frac{t}{\varepsilon^2}\right]) \quad \text{on} \quad \varepsilon\mathbb{Z}^d$$

by **the transition operator** T_ε:

$$T_\varepsilon f(x) = \sum_{y \in \varepsilon\mathbb{Z}^d} P^{(\varepsilon)}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) f(y), \quad f \in l_0^\infty(\varepsilon\mathbb{Z}^d).$$

The difference generator of the random walk $\hat{X}_\varepsilon(t)$ takes the form

$$L_\varepsilon \;=\; \frac{1}{\varepsilon^2}(T_\varepsilon - I).$$

Rescaled process

Let $\varepsilon\mathbb{Z}^d = \{x : \frac{x}{\varepsilon} \in \mathbb{Z}^d\}$ be a compression of the lattice $\mathbb{Z}^d$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon A^\sharp \cup \varepsilon B^\sharp,$$

and we define **the rescaled random walk**

$$\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}(\left[\frac{t}{\varepsilon^2}\right]) \quad \text{on} \quad \varepsilon\mathbb{Z}^d$$

by **the transition operator** T_ε:

$$T_\varepsilon f(x) = \sum_{y \in \varepsilon\mathbb{Z}^d} P^{(\varepsilon)}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) f(y), \quad f \in l_0^\infty(\varepsilon\mathbb{Z}^d).$$

The difference generator of the random walk $\hat{X}_\varepsilon(t)$ takes the form

$$L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon - I).$$

Rescaled process

Let $\varepsilon\mathbb{Z}^d = \{x : \frac{x}{\varepsilon} \in \mathbb{Z}^d\}$ be a compression of the lattice $\mathbb{Z}^d$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon A^\sharp \cup \varepsilon B^\sharp,$$

and we define **the rescaled random walk**

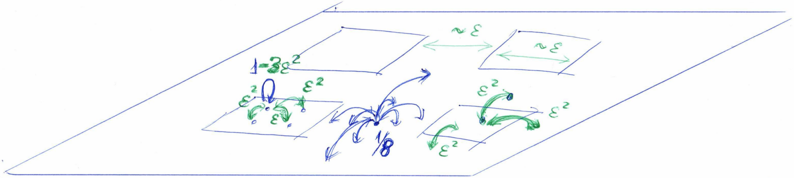
$$\widehat{X}_\varepsilon(t) = \varepsilon\widehat{X}(\left[\frac{t}{\varepsilon^2}\right]) \quad \text{on} \quad \varepsilon\mathbb{Z}^d$$

by **the transition operator** T_ε:

$$T_\varepsilon f(x) = \sum_{y\in\varepsilon\mathbb{Z}^d} P^{(\varepsilon)}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) f(y), \quad f \in l_0^\infty(\varepsilon\mathbb{Z}^d).$$

The difference generator of the random walk $\hat{X}_\varepsilon(t)$ takes the form

$$L_\varepsilon = \frac{1}{\varepsilon^2}(T_\varepsilon - I).$$



~ε
~ε
ε²
ε²
ε²
ε²
1/8
ε²
ε²

Limit behaviour under diffusive scaling

Our goal is to describe the large time behavior of the random walk $\widehat{X}_\varepsilon(t)$ and to construct the limit process.

Idea: In addition to the coordinate $\widehat{X}_\varepsilon(t)$ of the random walk on the lattice we introduce extra variables $k(\widehat{X}_\varepsilon(t))$ that characterizes the position of the random walk inside the period. Then the limit dynamics of this two-component process

$$X_\varepsilon(t) = \big(\widehat{X}_\varepsilon(t),\, k(\widehat{X}_\varepsilon(t))\big)$$

is Markovian.

The components of the limit process are coupled, thus the projection of the Markov process on the "spatial" component is not Markov any more.

Limit behaviour under diffusive scaling

Our goal is to describe the large time behavior of the random walk $\widehat{X}_\varepsilon(t)$ and to construct the limit process.

Idea: In addition to the coordinate $\widehat{X}_\varepsilon(t)$ of the random walk on the lattice we introduce extra variables $k(\widehat{X}_\varepsilon(t))$ that characterizes the position of the random walk inside the period. Then the limit dynamics of this two-component process

$$X_\varepsilon(t) = \big(\widehat{X}_\varepsilon(t),\, k(\widehat{X}_\varepsilon(t))\big)$$

is Markovian.

The components of the limit process are coupled, thus the projection of the Markov process on the "spatial" component is not Markov any more.

Limit behaviour under diffusive scaling

Our goal is to describe the large time behavior of the random walk $\widehat{X}_\varepsilon(t)$ and to construct the limit process.

Idea: In addition to the coordinate $\widehat{X}_\varepsilon(t)$ of the random walk on the lattice we introduce extra variables $k(\widehat{X}_\varepsilon(t))$ that characterizes the position of the random walk inside the period. Then the limit dynamics of this two-component process

$$X_\varepsilon(t) = \big(\widehat{X}_\varepsilon(t),\, k(\widehat{X}_\varepsilon(t))\big)$$

is Markovian.

The components of the limit process are coupled, thus the projection of the Markov process on the "spatial" component is not Markov any more.

Construction of the limit Markov semigroup $T(t)$

Assume that the set A contains $M \in \mathbb{N}$ sites of the periodicity cell:

$$A = \{x_1, \dots, x_M\}, \qquad M \geq 1.$$

We denote $E = \mathbb{R}^d \times \{0, 1, \dots, M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity.

A function $F = F(x, k) \in C_0(E)$ can be represented as a vector function

$$F(x, k) = \{f_k(x) \in C_0(\mathbb{R}^d),\ k = 0, 1, \dots, M\}.$$

The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max_{k=0,1\dots,M} \|f_k\|_{C_0(\mathbb{R}^d)}.$$

Construction of the limit Markov semigroup $T(t)$

Assume that the set A contains $M \in \mathbb{N}$ sites of the periodicity cell:

$$A = \{x_1, \ldots, x_M\}, \qquad M \geq 1.$$

We denote $E = \mathbb{R}^d \times \{0, 1, \ldots, M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity.

A function $F = F(x, k) \in C_0(E)$ can be represented as a vector function

$$F(x, k) = \{f_k(x) \in C_0(\mathbb{R}^d),\ k = 0, 1, \ldots, M\}.$$

The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max_{k=0,1\ldots,M} \|f_k\|_{C_0(\mathbb{R}^d)}.$$

Construction of the limit Markov semigroup $T(t)$

Assume that the set A contains $M \in \mathbb{N}$ sites of the periodicity cell:

$$A = \{x_1, \dots, x_M\}, \qquad M \geq 1.$$

We denote $E = \mathbb{R}^d \times \{0, 1, \dots, M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity.

A function $F = F(x, k) \in C_0(E)$ can be represented as a vector function

$$F(x, k) = \{f_k(x) \in C_0(\mathbb{R}^d),\ k = 0, 1, \dots, M\}.$$

The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max_{k=0,1\dots,M} \|f_k\|_{C_0(\mathbb{R}^d)}.$$

Construction of the limit Markov semigroup $T(t)$

Assume that the set A contains $M \in \mathbb{N}$ sites of the periodicity cell:

$$A = \{x_1, \dots, x_M\}, \qquad M \geq 1.$$

We denote $E = \mathbb{R}^d \times \{0, 1, \dots, M\}$, and $C_0(E)$ stands for the Banach space of continuous functions vanishing at infinity.

A function $F = F(x, k) \in C_0(E)$ can be represented as a vector function

$$F(x, k) = \{f_k(x) \in C_0(\mathbb{R}^d),\ k = 0, 1, \dots, M\}.$$

The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max_{k=0,1\dots,M} \|f_k\|_{C_0(\mathbb{R}^d)}.$$

The generator of the limit semigroup

Consider the operator

$$(LF)(x,k) = \begin{pmatrix} \Theta \cdot \nabla\nabla f_0(x) \\ 0 \\ \dots \\ 0 \end{pmatrix} + L_A F(x,k),$$

where Θ is a positive definite matrix (defined in terms of the homogenization problem), and L_A is a generator of a continuous time Markov jump process

$$L_A F(x,k) = \lambda(k) \sum_{\substack{j=0 \\ j \neq k}}^{M} \mu_{kj}(f_j(x) - f_k(x)).$$

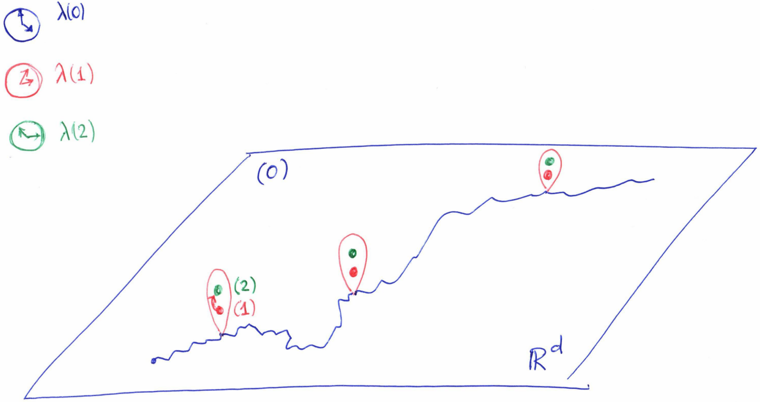
The intensities of jump rates are

$$\alpha_{0j} \ = \ \frac{1}{|B|} \sum_{y \in B} v(y, y_j), \quad \alpha_{j0} = \sum_{y \in B} v(y_j, y), \quad \alpha_{kj} = v(y_k, y_j).$$

$$\lambda(k) = \sum_{\substack{j=0 \\ j \neq k}}^{M} \alpha_{kj}, \quad \mu_{kj} = \frac{\alpha_{kj}}{\lambda(k)}, \quad j, k = 0, 1, \ldots, M, \quad j \neq k,$$

Remark. The coefficients of the operator L_A depend only on the elements of matrix V.

Astral diffusion



The semigroup

The operator L is defined on the core

$$D = \{(f_0, f_1, \dots, f_M),\ f_0 \in C_0^\infty(\mathbb{R}^d),\\ f_j \in C_0(\mathbb{R}^d),\ j = 1, \dots, M\}$$

which is a dense set in $C_0(E)$. The operator L on $C_0(E)$ satisfies the positive maximum principle, i.e. if $F \in C_0(E)$ and $\max_E F(x,k) = F(x_0,k_0) = f_{k_0}(x_0)$, then $LF(x_0,k_0) \le 0$.

Then by the Hille-Yosida theorem the closure of L is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$, that is a Feller semigroup.

Question: How to see the semigroup convergence

$$T_\varepsilon^{\left[\frac{t}{\varepsilon^2}\right]} \to T(t)?$$

The semigroup

The operator L is defined on the core

$$D = \{(f_0, f_1, \ldots, f_M),\ f_0 \in C_0^\infty(\mathbb{R}^d), \\ f_j \in C_0(\mathbb{R}^d),\ j = 1, \ldots, M\}$$

which is a dense set in $C_0(E)$. The operator L on $C_0(E)$ satisfies the positive maximum principle, i.e. if $F \in C_0(E)$ and $\max_E F(x,k) = F(x_0,k_0) = f_{k_0}(x_0)$, then $LF(x_0,k_0) \leq 0$.

Then by the Hille-Yosida theorem the closure of L is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$, that is a Feller semigroup.

Question: How to see the semigroup convergence

$$T_\varepsilon^{\left[\frac{t}{\varepsilon^2}\right]} \to T(t)?$$

The semigroup

The operator L is defined on the core

$$D = \{(f_0, f_1, \dots, f_M),\ f_0 \in C_0^\infty(\mathbb{R}^d),\\ f_j \in C_0(\mathbb{R}^d),\ j = 1, \dots, M\}$$

which is a dense set in $C_0(E)$. The operator L on $C_0(E)$ satisfies the positive maximum principle, i.e. if $F \in C_0(E)$ and $\max_E F(x,k) = F(x_0,k_0) = f_{k_0}(x_0)$, then $LF(x_0,k_0) \leq 0$.

Then by the Hille-Yosida theorem the closure of L is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$, that is a Feller semigroup.

Question: How to see the semigroup convergence

$$T_\varepsilon^{\left[\frac{t}{\varepsilon^2}\right]} \to T(t)?$$

Construction of the extended process

First we equip the random walk $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}([\frac{t}{\varepsilon^2}])$ with an additional component $k(\widehat{X}_\varepsilon(t))$. For the extended process $X_\varepsilon(t)$ we prove convergence of the corresponding semigroups.

The additional coordinates characterize the position of a random walker in the slow component.

If we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$ for each $k = 1, \dots, M$, then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \ldots \cup \varepsilon\{x_M\}^\sharp.$$

To each point $x \in \varepsilon\mathbb{Z}^d$ we assign an index $k(x) \in \{0, 1, \dots, M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} 0, & \text{if } x \in \varepsilon B^\sharp; \\ j, & \text{if } x \in \varepsilon\{x_j\}^\sharp, \ j = 1, \dots, M. \end{cases}$$

Construction of the extended process

First we equip the random walk $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}([\frac{t}{\varepsilon^2}])$ with an additional component $k(\widehat{X}_\varepsilon(t))$. For the extended process $X_\varepsilon(t)$ we prove convergence of the corresponding semigroups.

The additional coordinates characterize the position of a random walker in the slow component.

If we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$ for each $k = 1, \dots, M$, then

$$\varepsilon \mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B^\sharp \cup \varepsilon \{x_1\}^\sharp \cup \ldots \cup \varepsilon \{x_M\}^\sharp.$$

To each point $x \in \varepsilon \mathbb{Z}^d$ we assign an index $k(x) \in \{0, 1, \dots, M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} 0, & \text{if } x \in \varepsilon B^\sharp; \\ j, & \text{if } x \in \varepsilon \{x_j\}^\sharp, \ j = 1, \dots, M. \end{cases}$$

Construction of the extended process

First we equip the random walk $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}([\frac{t}{\varepsilon^2}])$ with an additional component $k(\widehat{X}_\varepsilon(t))$. For the extended process $X_\varepsilon(t)$ we prove convergence of the corresponding semigroups.

The additional coordinates characterize the position of a random walker in the slow component.

If we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$ for each $k = 1, \ldots, M$, then

$$\varepsilon \mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B^\sharp \cup \varepsilon \{x_1\}^\sharp \cup \ldots \cup \varepsilon \{x_M\}^\sharp.$$

To each point $x \in \varepsilon \mathbb{Z}^d$ we assign an index $k(x) \in \{0, 1, \ldots, M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} 0, & \text{if } x \in \varepsilon B^\sharp; \\ j, & \text{if } x \in \varepsilon \{x_j\}^\sharp,\ j = 1, \ldots, M. \end{cases}$$

Construction of the extended process

First we equip the random walk $\widehat{X}_\varepsilon(t) = \varepsilon \widehat{X}([\frac{t}{\varepsilon^2}])$ with an additional component $k(\widehat{X}_\varepsilon(t))$. For the extended process $X_\varepsilon(t)$ we prove convergence of the corresponding semigroups.

The additional coordinates characterize the position of a random walker in the slow component.

If we denote by $\{x_k\}^\sharp$ the periodic extension of the point $x_k \in A$ for each $k = 1, \dots, M$, then

$$\varepsilon \mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B^\sharp \cup \varepsilon \{x_1\}^\sharp \cup \ldots \cup \varepsilon \{x_M\}^\sharp.$$

To each point $x \in \varepsilon \mathbb{Z}^d$ we assign an index $k(x) \in \{0, 1, \dots, M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} 0, & \text{if } x \in \varepsilon B^\sharp; \\ j, & \text{if } x \in \varepsilon \{x_j\}^\sharp, \ j = 1, \dots, M. \end{cases}$$

The extended process: space and transition operator

Then we introduce the space

$$E_\varepsilon \;=\; \left\{(x,k(x)),\; x\in\varepsilon\mathbb{Z}^d,\; k(x)\in\{0,1,\ldots,M\}\right\},$$
$$E_\varepsilon\subset\varepsilon\mathbb{Z}^d\times\{0,1,\ldots,M\}.$$

We can take $x\in\varepsilon\mathbb{Z}^d$ as a coordinate on E_ε.

Let $\mathcal{B}(E_\varepsilon)$ be the space of bounded functions on E_ε and T_ε be the transition operator of the extended random walk

$$X_\varepsilon(t)=(\widehat{X}_\varepsilon(t),k(\widehat{X}_\varepsilon(t)))$$

on E_ε with the same transition probabilities of the random walk on $\varepsilon\mathbb{Z}^d$ as above:

$$(T_\varepsilon f)(x,k(x))=\sum_{y\in\varepsilon\mathbb{Z}^d}p_\varepsilon(x,y)f(y,k(y)),\quad f\in\mathcal{B}(E_\varepsilon).$$

The operator T_ε is a contraction on $\mathcal{B}(E_\varepsilon)$

The extended process: space and transition operator

Then we introduce the space

$$E_\varepsilon \;=\; \left\{(x,k(x)),\; x\in\varepsilon\mathbb{Z}^d,\; k(x)\in\{0,1,\dots,M\}\right\},$$
$$E_\varepsilon\subset\varepsilon\mathbb{Z}^d\times\{0,1,\dots,M\}.$$

We can take $x\in\varepsilon\mathbb{Z}^d$ as a coordinate on E_ε.

Let $\mathcal{B}(E_\varepsilon)$ be the space of bounded functions on E_ε and T_ε be the transition operator of the extended random walk

$$X_\varepsilon(t)=(\widehat{X}_\varepsilon(t),k(\widehat{X}_\varepsilon(t)))$$

on E_ε with the same transition probabilities of the random walk on $\varepsilon\mathbb{Z}^d$ as above:

$$(T_\varepsilon f)(x,k(x))=\sum_{y\in\varepsilon\mathbb{Z}^d}p_\varepsilon(x,y)f(y,k(y)),\quad f\in\mathcal{B}(E_\varepsilon).$$

The operator T_ε is a contraction on $\mathcal{B}(E_\varepsilon)$

The extended process: space and transition operator

Then we introduce the space

$$E_\varepsilon \;=\; \left\{(x,k(x)),\; x\in\varepsilon\mathbb{Z}^d,\; k(x)\in\{0,1,\ldots,M\}\right\},$$
$$E_\varepsilon \subset \varepsilon\mathbb{Z}^d\times\{0,1,\ldots,M\}.$$

We can take $x\in\varepsilon\mathbb{Z}^d$ as a coordinate on E_ε.

Let $\mathcal{B}(E_\varepsilon)$ be the space of bounded functions on E_ε and T_ε be the transition operator of the extended random walk

$$X_\varepsilon(t) = (\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$$

on E_ε with the same transition probabilities of the random walk on $\varepsilon\mathbb{Z}^d$ as above:

$$(T_\varepsilon f)(x,k(x)) = \sum_{y\in\varepsilon\mathbb{Z}^d} p_\varepsilon(x,y) f(y,k(y)), \quad f\in\mathcal{B}(E_\varepsilon).$$

The operator T_ε is a contraction on $\mathcal{B}(E_\varepsilon)$

The projection operator $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$

Let $l_0^\infty(E_\varepsilon)$ be a Banach space of bounded functions on E_ε vanishing as $|x| \to \infty$ with the norm

$$\|f\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |f(x,k(x))| = \sup_{x\in\varepsilon\mathbb{Z}^d} |f(x,k(x))|.$$

For every $F \in C_0(E)$ we define on E_ε the function $\pi_\varepsilon F \in l_0^\infty(E_\varepsilon)$ as follows:

$$(\pi_\varepsilon F)(x,k(x)) = \begin{cases} f_0(x), & \text{if } x \in \varepsilon B^\sharp,\ k(x)=0; \\ f_1(x), & \text{if } x \in \varepsilon\{x_1\}^\sharp,\ k(x)=1; \\ \dots & \\ f_M(x), & \text{if } x \in \varepsilon\{x_M\}^\sharp,\ k(x)=M. \end{cases}$$

Then π_ε defines a **bounded** linear transformation $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$:

$$\|\pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |(\pi_\varepsilon F)(x,k(x))| \le \|F\|_{C_0(E)}, \qquad \sup_\varepsilon \|\pi_\varepsilon\| \le 1.$$

The projection operator $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$

Let $l_0^\infty(E_\varepsilon)$ be a Banach space of bounded functions on E_ε vanishing as $|x| \to \infty$ with the norm

$$\|f\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |f(x,k(x))| = \sup_{x\in\varepsilon\mathbb{Z}^d} |f(x,k(x))|.$$

For every $F \in C_0(E)$ we define on E_ε the function $\pi_\varepsilon F \in l_0^\infty(E_\varepsilon)$ as follows:

$$(\pi_\varepsilon F)(x,k(x)) = \begin{cases} f_0(x), & \text{if } x \in \varepsilon B^\sharp,\ k(x) = 0; \\ f_1(x), & \text{if } x \in \varepsilon\{x_1\}^\sharp,\ k(x) = 1; \\ \dots & \\ f_M(x), & \text{if } x \in \varepsilon\{x_M\}^\sharp,\ k(x) = M. \end{cases}$$

Then π_ε defines a **bounded** linear transformation $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$:

$$\|\pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |(\pi_\varepsilon F)(x,k(x))| \le \|F\|_{C_0(E)}, \qquad \sup_\varepsilon \|\pi_\varepsilon\| \le 1.$$

The projection operator $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$

Let $l_0^\infty(E_\varepsilon)$ be a Banach space of bounded functions on E_ε vanishing as $|x| \to \infty$ with the norm

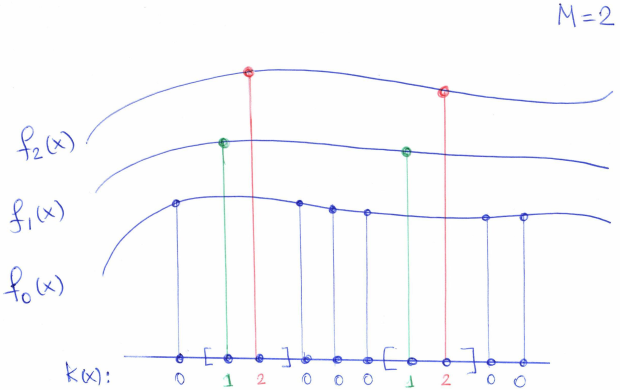
$$\|f\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |f(x,k(x))| = \sup_{x\in\varepsilon\mathbb{Z}^d} |f(x,k(x))|.$$

For every $F \in C_0(E)$ we define on E_ε the function $\pi_\varepsilon F \in l_0^\infty(E_\varepsilon)$ as follows:

$$(\pi_\varepsilon F)(x,k(x)) \;=\; \begin{cases} f_0(x), & \text{if } x \in \varepsilon B^\sharp,\ k(x)=0; \\ f_1(x), & \text{if } x \in \varepsilon\{x_1\}^\sharp,\ k(x)=1; \\ \dots & \\ f_M(x), & \text{if } x \in \varepsilon\{x_M\}^\sharp,\ k(x)=M. \end{cases}$$

Then π_ε defines a **bounded** linear transformation $\pi_\varepsilon : C_0(E) \to l_0^\infty(E_\varepsilon)$:

$$\|\pi_\varepsilon F\|_{l_0^\infty(E_\varepsilon)} = \sup_{(x,k(x))\in E_\varepsilon} |(\pi_\varepsilon F)(x,k(x))| \le \|F\|_{C_0(E)}, \qquad \sup_\varepsilon \|\pi_\varepsilon\| \le 1.$$


M=2
f2(x)
f1(x)
f0(x)
k(x):
0 1 2 0 0 0 1 2 0 0

The semigroup convergence

Theorem

Let $T(t)$ *be a strongly continuous, positive, contraction semigroup on* $C_0(E)$ *with generator* L *defined by*

$$(LF)(x,k) = \begin{pmatrix} \Theta \cdot \nabla\nabla f_0(x) \\ 0 \\ \dots \\ 0 \end{pmatrix} + L_A F(x,k),$$

and T_ε *be the linear operator on* $l_0^\infty(E_\varepsilon)$ *defined above (the transition operator of the extended random walk* $X_\varepsilon(t) = (\widehat{X}_\varepsilon(t), k(\widehat{X}_\varepsilon(t)))$ *on* E_ε*).*

Then for every $F \in C_0(E)$

$$T_\varepsilon^{\left[\frac{t}{\varepsilon^2}\right]} \pi_\varepsilon F \rightarrow T(t)F \quad \textit{for all} \quad t \geq 0 \tag{1}$$

as $\varepsilon \to 0$*.*

The idea of the proof

The proof of the Theorem relies on the following approximation theorem

Theorem (Theorem 6.5, Ch.1, S. N. Ethier, T. G. Kurtz, Markov processes: Characterization and convergence, 2005.)

For $n = 1, 2, \ldots$, let T_n be a linear contraction on the Banach space $\mathcal{L}_n$, let ε_n be a positive number, and put $A_n = \varepsilon_n^{-1}(T_n - E)$. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$.
Let $\{T(t)\}$ be a strongly continuous contraction semigroup on the Banach space $\mathcal{L}$ with generator A, and let D be a core for A.
Assume that $\pi_n : \mathcal{L} \to \mathcal{L}_n$ are bounded linear transformations with $\sup_n \|\pi_n\| < \infty$.
Then the following are equivalent:

a) For each $f \in \mathcal{L}$, $T_n^{\left[\frac{t}{\varepsilon_n}\right]}\pi_n f \ \to \ T(t)f$ for all $t \geq 0$ as $\varepsilon \to 0$.

b) For each $f \in D$, there exists $f_n \in \mathcal{L}_n$ for each $n \geq 1$ such that $f_n \to f$ and $A_n f_n \to Af$.

For every $F = (f_0, f_1, \dots, f_M) \in D$ we construct $F_\varepsilon \in l_0^\infty(E_\varepsilon)$ as a small perturbation of $\pi_\varepsilon F$:

$$F_\varepsilon = \pi_\varepsilon F + G_\varepsilon, \quad \|G_\varepsilon\|_{l_0^\infty(E_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

We consider the following $F_\varepsilon \in l_0^\infty(E_\varepsilon)$

$$F_\varepsilon(x, k(x)) = \begin{cases} f_0(x) + \varepsilon(\nabla f_0(x), h(\frac{x}{\varepsilon})) + \varepsilon^2(\nabla\nabla f_0(x), g(\frac{x}{\varepsilon})) \\ \qquad +\varepsilon^2 \sum_{j=1}^M q_j(\frac{x}{\varepsilon})(f_0(x) - f_j(x)), \\ \qquad \text{if } x \in \varepsilon B^\sharp,\ k(x) = 0, \\ f_1(x), \quad \text{if } x \in \varepsilon\{x_1\}^\sharp,\ k(x) = 1, \\ \dots \\ f_M(x), \quad \text{if } x \in \varepsilon\{x_M\}^\sharp,\ k(x) = M. \end{cases}$$

Here $h(y), g(y), q_j(y), j = 1, \dots, M$, are periodic bounded functions (correctors).

For every $F=(f_0,f_1,\dots,f_M)\in D$ we construct $F_\varepsilon\in l_0^\infty(E_\varepsilon)$ as a small perturbation of $\pi_\varepsilon F$:

$$F_\varepsilon=\pi_\varepsilon F+G_\varepsilon,\quad \|G_\varepsilon\|_{l_0^\infty(E_\varepsilon)}\to 0\quad \text{as } \varepsilon\to 0.$$

We consider the following $F_\varepsilon\in l_0^\infty(E_\varepsilon)$

$$F_\varepsilon(x,k(x)) \;=\; \begin{cases} f_0(x)+\varepsilon(\nabla f_0(x),h(\tfrac{x}{\varepsilon}))+\varepsilon^2(\nabla\nabla f_0(x),g(\tfrac{x}{\varepsilon})) \\ \qquad +\varepsilon^2\sum_{j=1}^M q_j(\tfrac{x}{\varepsilon})(f_0(x)-f_j(x)), \\ \qquad\quad \text{if } x\in\varepsilon B^\sharp,\ k(x)=0, \\ f_1(x),\quad \text{if } x\in\varepsilon\{x_1\}^\sharp,\ k(x)=1, \\ \dots \\ f_M(x),\quad \text{if } x\in\varepsilon\{x_M\}^\sharp,\ k(x)=M. \end{cases}$$

Here $h(y),g(y),q_j(y),j=1,\dots,M,$ are periodic bounded functions (correctors).

Correctors

Lemma

There exist bounded periodic functions $h(y) = \{h_i(y)\}_{i=1}^d$ *and* $g(y) = \{g_{im}(y)\}_{i,m=1}^d$ *(correctors) and a positive definite matrix* $\Theta > 0$*, such that the limit relation* $L_\varepsilon F_\varepsilon \to LF$ *holds for every* $F \in D$*.*

The matrix Θ *defined by*

$$\Theta = \frac{1}{|B|} \sum_{y \in B} \sum_{\xi \in \Lambda_y} p_0(y, y+\xi)\, \xi \otimes \left(\frac{1}{2}\xi + h(y+\xi) \right)$$

is positive definite, i.e. $(\Theta\eta, \eta) > 0 \quad \forall \eta \neq 0$*.*

Invariance principle. The limit Markov process

Thus we justified the convergence of the semigroups, and consequently, the convergence of finite dimensional distributions of $X_\varepsilon(t)$.
The next question is about existence of the limit process $\mathcal{X}(t)$ in E and convergence in the Skorokhod topology of $D_E[0,\infty)$.

Theorem (Invariance principle for the extended processes $X_\varepsilon(t)$)

For any initial distribution $\nu \in \mathcal{P}(E)$ there exists a Markov process $\mathcal{X}(t)$ corresponding to the semigroup $T(t) : C_0(E) \to C_0(E)$ with our generator L and with sample paths in $D_E[0,\infty)$.
If ν is the limit law of $X_\varepsilon(0)$, then

$$X_\varepsilon(t) \ \Rightarrow \ \mathcal{X}(t) \quad \textit{in} \ \ D_E[0,\infty) \ \ \textit{as} \ \ \varepsilon \to 0.$$

Invariance principle. The limit Markov process

Thus we justified the convergence of the semigroups, and consequently, the convergence of finite dimensional distributions of $X_\varepsilon(t)$.
The next question is about existence of the limit process $\mathcal{X}(t)$ in E and convergence in the Skorokhod topology of $D_E[0,\infty)$.

Theorem (Invariance principle for the extended processes $X_\varepsilon(t)$)

For any initial distribution $\nu \in \mathcal{P}(E)$ there exists a Markov process $\mathcal{X}(t)$ corresponding to the semigroup $T(t) : C_0(E) \to C_0(E)$ with our generator L and with sample paths in $D_E[0,\infty)$.
If ν is the limit law of $X_\varepsilon(0)$, then

$$X_\varepsilon(t) \ \Rightarrow \ \mathcal{X}(t) \quad \textit{in} \ \ D_E[0,\infty) \ \ \textit{as} \ \ \varepsilon \to 0.$$

Generalization: several fast components

We keep the assumptions on $P^{(\varepsilon)}(x,y)$, in particular we assume that the transition probabilities are periodic, have a finite range of interaction and define an irreducible random walk.

however, now we assume that $B^\sharp$ is the union of N, $N>1$, non-intersecting unbounded sets such that P^0 is periodic, invariant and irreducible on each of these sets.

We denote these sets $B_1^\sharp, \dots, B_N^\sharp$. Then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B_1^\sharp \cup \ldots \cup \varepsilon B_N^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \ldots \cup \varepsilon\{x_M\}^\sharp.$$

We assign to each $x \in \varepsilon\mathbb{Z}^d$ an index $k(x) \in \{1,\dots,N+M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} j, & \text{if } x \in \varepsilon B_j^\sharp,\ j=1,\dots,N; \\ N+j, & \text{if } x \in \varepsilon\{x_j\}^\sharp,\ j=1,\dots,M. \end{cases}$$

Generalization: several fast components

We keep the assumptions on $P^{(\varepsilon)}(x,y)$, in particular we assume that the transition probabilities are periodic, have a finite range of interaction and define an irreducible random walk.
however, now we assume that $B^\sharp$ is the union of N, $N > 1$, non-intersecting unbounded sets such that P^0 is periodic, invariant and irreducible on each of these sets.

We denote these sets $B_1^\sharp, \ldots, B_N^\sharp$. Then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B_1^\sharp \cup \ldots \cup \varepsilon B_N^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \ldots \cup \varepsilon\{x_M\}^\sharp.$$

We assign to each $x \in \varepsilon\mathbb{Z}^d$ an index $k(x) \in \{1, \ldots, N+M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} j, & \text{if } x \in \varepsilon B_j^\sharp, \ j = 1, \ldots, N; \\ N+j, & \text{if } x \in \varepsilon\{x_j\}^\sharp, \ j = 1, \ldots, M. \end{cases}$$

Generalization: several fast components

We keep the assumptions on $P^{(\varepsilon)}(x,y)$, in particular we assume that the transition probabilities are periodic, have a finite range of interaction and define an irreducible random walk.
however, now we assume that $B^\sharp$ is the union of N, $N>1$, non-intersecting unbounded sets such that P^0 is periodic, invariant and irreducible on each of these sets.

We denote these sets $B_1^\sharp,\dots,B_N^\sharp$. Then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B_1^\sharp \cup \ldots \cup \varepsilon B_N^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \ldots \cup \varepsilon\{x_M\}^\sharp.$$

We assign to each $x\in\varepsilon\mathbb{Z}^d$ an index $k(x)\in\{1,\dots,N+M\}$ depending on the component to which x belongs:

$$k(x) = \begin{cases} j, & \text{if } x\in\varepsilon B_j^\sharp,\ j=1,\dots,N; \\ N+j, & \text{if } x\in\varepsilon\{x_j\}^\sharp,\ j=1,\dots,M. \end{cases}$$

Generalization: several fast components

We keep the assumptions on $P^{(\varepsilon)}(x,y)$, in particular we assume that the transition probabilities are periodic, have a finite range of interaction and define an irreducible random walk.
however, now we assume that $B^\sharp$ is the union of N, $N > 1$, non-intersecting unbounded sets such that P^0 is periodic, invariant and irreducible on each of these sets.

We denote these sets $B_1^\sharp, \dots, B_N^\sharp$. Then

$$\varepsilon\mathbb{Z}^d = \varepsilon B^\sharp \cup \varepsilon A^\sharp = \varepsilon B_1^\sharp \cup \ldots \cup \varepsilon B_N^\sharp \cup \varepsilon\{x_1\}^\sharp \cup \ldots \cup \varepsilon\{x_M\}^\sharp.$$

We assign to each $x \in \varepsilon\mathbb{Z}^d$ an index $k(x) \in \{1, \dots, N+M\}$ depending on the component to which x belongs:

$$k(x) \; = \; \begin{cases} j, \;\; \text{if } \; x \in \varepsilon B_j^\sharp, \; j = 1, \dots, N; \\ N + j, \;\; \text{if } \; x \in \varepsilon\{x_j\}^\sharp, \; j = 1, \dots, M. \end{cases}$$

Then the limit Markov process has the similar structure and the generator L has the form

$$(LF)(x,k) = \begin{pmatrix} \Theta^1 \cdot \nabla\nabla f_1(x) \\ \dots \\ \Theta^N \cdot \nabla\nabla f_N(x) \\ 0 \\ \dots \\ 0 \end{pmatrix} + L_A F(x,k),$$

where $\Theta^1, \dots, \Theta^N$ are positive definite matrices, and L_A is a generator of a continuous time Markov jump process

$$L_A F(x,k) = \lambda(k) \sum_{\substack{j=1 \\ j \neq k}}^{N+M} \mu_{kj} (f_j(x) - f_k(x))$$

with jump rates $\lambda(k)\mu_{kj}$.

Evolution of the first component

Question: *How to describe an evolution for the first* **(spatial)** *component in the astral diffusion?*

Let us consider the case of an one-point astral set: $|A| = 1$.

Then $P(x,t) = (p_0(x,t), p_1(x,t))$, and let $(\pi_0(x), \pi_1(x))$ be the initial condition.
The evolution equation for $P(x,t)$ is

$$\begin{cases} \partial_t p_0 = \Theta \cdot \nabla\nabla p_0 - \lambda(0)p_0 + \lambda(1)p_1 \\ \partial_t p_1 = -\lambda(1)p_1 + \lambda(0)p_0 \end{cases}$$

Evolution of the first component

Question: *How to describe an evolution for the first* **(spatial)** *component in the astral diffusion?*

Let us consider the case of an one-point astral set: $|A| = 1$.

Then $P(x,t) = (p_0(x,t), p_1(x,t))$, and let $(\pi_0(x), \pi_1(x))$ be the initial condition.
The evolution equation for $P(x,t)$ is

$$\begin{cases} \partial_t p_0 = \Theta \cdot \nabla\nabla p_0 - \lambda(0) p_0 + \lambda(1) p_1 \\ \partial_t p_1 = -\lambda(1) p_1 + \lambda(0) p_0 \end{cases}$$

The solution of the second equation is

$$p_1(x,t) = e^{-\lambda(1)t}\pi_1(x) + \lambda(0)\int_0^t e^{-\lambda(1)(t-s)}p_0(x,s)ds,$$

where $\pi_1(x) = p_1(x,0)$.

Substitution of this solution into the first equation gives the following evolution equation on p_0:

$$\begin{cases} \partial_t p_0 = \Theta \cdot \nabla\nabla p_0 - \\ \qquad -\lambda(0)p_0 + \lambda(0)\lambda(1)\int_0^t e^{-\lambda(1)(t-s)}p_0(x,s)ds + \lambda(1)e^{-\lambda(1)t}\pi_1(x), \\ p_0(x,0) = \pi_0(x). \end{cases}$$

The solution of the second equation is

$$p_1(x,t) = e^{-\lambda(1)t}\pi_1(x) + \lambda(0)\int\limits_0^t e^{-\lambda(1)(t-s)}p_0(x,s)ds,$$

where $\pi_1(x) = p_1(x,0)$.

Substitution of this solution into the first equation gives the following evolution equation on p_0:

$$\begin{cases} \partial_t p_0 = \Theta \cdot \nabla\nabla p_0 - \\ \qquad -\lambda(0)p_0 + \lambda(0)\lambda(1)\int\limits_0^t e^{-\lambda(1)(t-s)}p_0(x,s)ds + \lambda(1)e^{-\lambda(1)t}\pi_1(x), \\ p_0(x,0) = \pi_0(x). \end{cases}$$

For the construction of the limit process for diffusions in high-contrast periodic media see

A. Piatnitski, S. Pirogov, E. Zhizhina, Limit behaviour of diffusion in high-contrast periodic media and related Markov semigroups, *Applicable Analysis*, **98**(1-2) (2019).

Denote $E = \mathbb{R}^d \times G^\star$, where $G^\star = G \cup \{\star\}$, then a function $F \in C_0(E)$ can be written in a vector form

$$F(x, \hat{y}) = (f_0(x),\ f_1(x, y)), \quad x \in \mathbb{R}^d,\ \hat{y} \in G^\star,\ y \in G$$

with $f_0 \in C_0(\mathbb{R}^d),\ f_1 \in C_0(\mathbb{R}^d, C(\overline{G}))$.

Denote by $C_0^G(E)$ a linear closed subspace of functions from $C_0(E)$ such that

$$f_1(x, y)|_{y \in \partial G} = f_0(x) \quad \forall x \in \mathbb{R}^d.$$

Denote $E = \mathbb{R}^d \times G^\star$, where $G^\star = G \cup \{\star\}$, then a function $F \in C_0(E)$ can be written in a vector form

$$F(x, \hat{y}) = (f_0(x),\ f_1(x, y)), \quad x \in \mathbb{R}^d,\ \hat{y} \in G^\star,\ y \in G$$

with $f_0 \in C_0(\mathbb{R}^d),\ f_1 \in C_0(\mathbb{R}^d, C(\overline{G}))$.

Denote by $C_0^G(E)$ a linear closed subspace of functions from $C_0(E)$ such that

$$f_1(x, y)|_{y \in \partial G} = f_0(x) \quad \forall x \in \mathbb{R}^d.$$

Let us consider in $C_0^G(E)$ an unbounded operator of the following form

$$(AF)(x,\hat{y}) = \begin{pmatrix} \Theta\nabla\nabla f_0(x) - \frac{1}{|G^c|}\int\limits_G \triangle_y f_1(x,y)dy \\ \\ \triangle_y f_1(x,y) \end{pmatrix}.$$

The domain $D(A)$ of the operator A is the closure (in the graph norm) of

$$D_A = \Big\{u_0 \in C_0^\infty(\mathbb{R}^d),\ u_1 \in C_0^\infty(\mathbb{R}^d;\ C^\infty(\overline{G})),\ u_1(x,y)|_{y\in\partial G} = u_0(x),$$
$$\triangle_y u_1(x,y)\Big|_{y\in\partial G} = \Theta\nabla\nabla u_0(x) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial u_1(x,y)}{\partial n_y^-} d\sigma(y)\Big\}.$$

Lemma

The closure of the operator A is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0^G(E)$.

Let us consider in $C_0^G(E)$ an unbounded operator of the following form

$$(AF)(x,\hat{y}) = \left(\begin{array}{c} \Theta\nabla\nabla f_0(x) - \frac{1}{|G^c|} \int\limits_G \triangle_y f_1(x,y)dy \\ \\ \triangle_y f_1(x,y) \end{array} \right).$$

The domain $D(A)$ of the operator A is the closure (in the graph norm) of

$$D_A = \Big\{ u_0 \in C_0^\infty(\mathbb{R}^d),\ u_1 \in C_0^\infty(\mathbb{R}^d;\ C^\infty(\overline{G})),\ u_1(x,y)|_{y\in\partial G} = u_0(x),$$

$$\triangle_y u_1(x,y)\Big|_{y\in\partial G} = \Theta\nabla\nabla u_0(x) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial u_1(x,y)}{\partial n_y^-} d\sigma(y) \Big\}.$$

Lemma

The closure of the operator A is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0^G(E)$.

Let us consider in $C_0^G(E)$ an unbounded operator of the following form

$$(AF)(x,\hat{y}) = \left(\begin{array}{c} \Theta\nabla\nabla f_0(x) - \frac{1}{|G^c|}\int\limits_G \triangle_y f_1(x,y)dy \\ \\ \triangle_y f_1(x,y) \end{array} \right).$$

The domain $D(A)$ of the operator A is the closure (in the graph norm) of

$$D_A = \Big\{ u_0 \in C_0^\infty(\mathbb{R}^d),\ u_1 \in C_0^\infty(\mathbb{R}^d;\ C^\infty(\overline{G})),\ u_1(x,y)|_{y\in\partial G} = u_0(x),$$

$$\triangle_y u_1(x,y)\Big|_{y\in\partial G} = \Theta\nabla\nabla u_0(x) + \frac{1}{|G^c|}\int\limits_{\partial G} \frac{\partial u_1(x,y)}{\partial n_y^-} d\sigma(y) \Big\}.$$

Lemma

The closure of the operator A is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0^G(E)$.

Thank you for your attention!