Philosophy of Natural Numbers (mathematical speculation)

Yuri Kondratiev

Bielefeld University
Dragomanov University, Kyiv

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speculation =

thinking
meditation
reflection
thought
contemplation

Plato theory

$$\text{Plato} = \Pi\lambda\alpha\tau\omega\nu = \text{Platon}$$

- Real world of ideas
- Real world of things
- Myth about the cavern
- From ideas to shades (reflections) as things

The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is a fundamental object in the mathematics. In certain sense $\mathbb{N}$ is the root of all modern mathematics. Other mathematical structures may be created as a logical development of this object.

The latter motivated L. Kronecker who summarized "God made the integers, all else is the work of man".

I.Kant: "Two things fill the mind: the starry heavens above me and the moral law within me".

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A number $n \in \mathbb{N}$ we interpret as a number of objects (a population) located in a location space X. For simplicity we take $X = \mathbb{R}^d$. The collection of all n-point subsets (or configurations with n elements) form a locally compact space $\Gamma^{(n)}(\mathbb{R}^d)$. It is the space (quite huge) of ideas for the number n.

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$$\Gamma_0(\mathbb{R}^d) = \cup_{n=0}^{\infty} \Gamma^{(n)}(\mathbb{R}^d)$$

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We can consider additionally the set $\Gamma(\mathbb{R}^d)$ consisting all locally finite configurations. This set may be considered as the space of ideas which corresponds to natural numbers and additionally to the actual infinity which is absent in the classical framework on natural numbers.

In such extension of $\mathbb{N}$ we arrive in the main question. Namely, most important mathematical theories related to natural numbers we need to develop to this new level. It concerns, first of all, the combinatorics that play central role in many mathematical structures and applications from probability theory to genetics.

Classical combinatoric

The combinatoric is dealing with the set of natural numbers $\mathbb{N}$ and relations between them.
Binomial coefficients:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

defined for $n \in \mathbb{N}$ and $0 \leq k \leq n$. Introducing the falling factorial $(n)_k$ we can write

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

These coefficients may be extended using embedding $\mathbb{N} \subset \mathbb{R}$ to polynomials

$$N_k(t) := \binom{t}{k} = \frac{t(t-1)\dots(t-k+1)}{k!} = \frac{(t)_k}{k!},\ t \in \mathbb{R}$$

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Chu-Vandermond relations:

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Generation functions

An alternative definition is given by the generation function

$$e_\lambda(t) := e^{t\log(1+\lambda)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(t)_n = \sum_{n=)}^{\infty} \lambda^n N_n(t).$$

Such transition to continuous variables makes possible to apply in discrete mathematics methods of analysis.

Using many particular GF we may create different polynomial systems.

Combinatoric and difference calculus

Transition to continuous variables makes possible to apply in discrete mathematics methods of analysis. In particular, let us define for functions $f : \mathbb{R} \to \mathbb{R}$ difference operators

$$(D^+ f)(t) = f(t+1) - f(t),$$
$$(D^- f)(t) = f(t-1) - f(t).$$

By a direct computation we obtain

$$D^+ (t)_n = n\ (t)_{n-1},$$
$$D^- (t)_n = -n\ (t-1)_{n-1}.$$

Additionally,

$$D^+ e_\lambda(t) = \lambda e_\lambda(t).$$

In this way we arrive in the framework of difference calculus closely related with the combinatoric.

There are specific questions inside difference calculus as, e.g., an analysis of Newton series

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K-transform

For functions $a : \mathbb{N} \to \mathbb{R}$ we define $b : \mathbb{N} \to \mathbb{R}$ as

$$b = Ka, \;\; b(n) = \sum_{k=0}^{n} \binom{n}{k} a(k).$$

This operator K (aka combinatorial transform) is very useful in combinatoric and its inverse gives so-called inclusion-exclusion formula:

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b(k).$$

Note that for $a : \mathbb{N} \to \mathbb{R}, \;\; a(j) = 0, j \neq k, a(k) = 1$

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Towards continuum

Any $n \in \mathbb{N}$ we interpret as the size of a population. It is convenient in the study of population models. There is a natural generalization leading to spatial ecological models. Now we would like to consider objects located in a given locally compact space X. For simplicity we will work with the Euclidean space $\mathbb{R}^d$.

For the substitution of $\mathbb{N}$ in this situation we can use two possible sets. Denote $\Gamma(\mathbb{R}^d)$ the set of all locally finite configurations (subsets) from $\mathbb{R}^d$.

$$\Gamma(\mathbb{R}^d) = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty, \text{any compact } K \subset \mathbb{R}^d\}.$$

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Finite configurations

Another possibility, is to introduce the set of all finite configurations $\Gamma_0(\mathbb{R}^d)$. Then

$$\Gamma_0(\mathbb{R}^d) = \cup_{n=0}^{\infty} \Gamma^{(n)}(\mathbb{R}^d),$$

where $\Gamma^{(n)}(\mathbb{R}^d)$ denoted the set of all configurations with n elements. We will see that in the continuous combinatoric the spaces $\Gamma(\mathbb{R}^d)$ and $\Gamma_0(\mathbb{R}^d)$ will play very different roles. It is a specific moment related with transition to the continuum. In this sense $\mathbb{N}$ is splitting in these two spaces of configurations.

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Note from the beginning, that the analog of the extension $\mathbb{N} \subset \mathbb{R}$ now naturally play the pair $\Gamma(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ where we have in mind an imbedding of configurations in the space of discrete Radon measures on $\mathbb{R}^d$ and, as a result, in the space of all Radon mesures on $\mathbb{R}^d$:

$$\Gamma(X) \ni \gamma \mapsto \gamma(dx) = \sum_{y\in\gamma} \delta_y(dx) \in \mathcal{M}(\mathbb{R}^d).$$

Therefore, instead of pair

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In spatial combinatoric main objects will be measure-valued

Falling factorials and Newton polynomials

For a test function $0 \le \xi \in \mathcal{D}(\mathbb{R}^d)$ consider a function

$$E_\xi(\omega) = e^{<\ln(1+\xi),\omega>} \quad \omega \in \mathcal{D}'(\mathbb{R}^d).$$

The power decomposition w.r.t. ξ gives

$$E_\xi(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} < \xi^{\otimes n}, (\omega)_n > .$$

Generalized kernels $(\omega)_n \in \mathcal{D}'(\mathbb{R}^{nd})$ are called infinite dimensional falling factorials on $\mathcal{D}'(\mathbb{R}^d)$. Define binomial coefficients (Newton polynomials) on $\mathcal{D}'(\mathbb{R}^d)$ as

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Infinite dimensional Chu-Vandermond relations on configurations:

$$(\gamma_1 \cup \gamma_2)_n = \sum_{k=0}^{n} \binom{n}{k} (\gamma_1)_k \otimes (\gamma_2)_{n-k}.$$

Theorem

For $\omega \in \mathcal{M}(\mathbb{R}^d)$

$$(\omega)_0 = 1$$

$$(\omega)_1 = \omega$$

$$(\omega)_n(x_1, \dots, x_n) = \omega(x_1)(\omega(x_2) - \delta_{x_1}(x_2)) \dots (\omega(x_n) - \delta_{x_1}(x_n) - \dots - \delta_{x_{n-1}}(x$$

In the particular case $\omega = \gamma = \{x_i \mid i \in \mathbb{N}\}$

$$(\gamma)_n = n! \binom{\gamma}{n} = \sum_{\{i_1, \dots, i_n\} \subset \mathbb{N}} \delta_{x_1} \odot \dots \odot \delta_{x_n},$$

where $\delta_{x_1} \odot \dots \odot \delta_{x_n}$ denotes symmetric tensor product.

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Falling factorials as measures

We have

$$\Gamma(\mathbb{R}^d) \ni \gamma \mapsto \gamma(dx) \in \mathcal{M}(\mathbb{R}^d).$$

Due to our construction

$$(\gamma)_n \in \mathcal{M}(\mathbb{R}^{nd})$$

is a symmetric Radon masure. Therefore, we arrive in measure valued Newton polynomials.

The latter is the main consequence of continuous combinatoric transition.

Difference geometry

For any $x \in \gamma$ define an elementary Markov death operator (death gradient)

$$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$$

and the tangent space $T_\gamma^-(\Gamma) = L^2(\mathbb{R}^d, \gamma)$. Then for $\psi \in C_0(\mathbb{R}^d)$

$$D_\psi^- F(\gamma) = \sum_{x \in \gamma} \psi(x)(F(\gamma \setminus x) - F(\gamma))$$

is the directional (difference) derivative.

Similarly, we define for $x \in \mathbb{R}^d$

$$D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma)$$

and the tangent space $T_\gamma^-(\Gamma) = L^2(\mathbb{R}^d, dx)$. Then for $\varphi \in C_0(\mathbb{R}^d)$

$$D_\varphi^+ F(\gamma) = \int_{\mathbb{R}^d} \varphi(x)(F(\gamma \cup x) - F(\gamma))dx$$

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For $\varphi \in C_0(\mathbb{R}^d)$ define a function

$$E_\varphi(\gamma) = \exp(< \gamma, \log(1+\varphi) >), \quad \gamma \in \Gamma.$$

It is the GF for the system on falling factorials (Newton polynomials) on Γ:

$$E_\varphi(\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} < \varphi^{\otimes n}, (\gamma)_n > .$$

Then

$$D^+_\psi E_\varphi(\gamma) =< \varphi\psi > E_\varphi(\gamma).$$

An explicit formula for the falling factorials (as measures on $(\mathbb{R}^d)^n$) is

$$(\gamma)_n = \sum_{\{x_1,\dots,x_n\}\subset\gamma} \delta_{x_1} \odot \delta_{x_2} \odot \cdots \odot \delta_{x_n},$$

where $\delta_{x_1} \odot \delta_{x_2} \odot \cdots \odot \delta_{x_n}$ denotes the symmetric tensor product of measures.

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The action of difference derivatives on Newton monomials is given by

$$D^+_\psi < \varphi^{(n)}, (\gamma)_n >= n \int_{\mathbb{R}^d} \psi(x) < \varphi^{(n)}(x, \cdot), (\gamma)_{n-1}(\cdot) > dx,$$

$$D^-_\psi < \varphi^{(n)}, (\gamma)_n >= -n \sum_{x \in \gamma} \psi(x) < \varphi^{(n)}(x, \cdot), (\gamma \setminus x)_{n-1}(\cdot) > .$$

Stirling kernels

We have polynomial equality

$$(\gamma)_n = \sum_{k=1}^{n} s_k^n \gamma^{\otimes k},$$

where

$$s_k^n : \mathcal{D}'(\mathbb{R}^{kd}) \to \mathcal{D}'(\mathbb{R}^{nd})$$

is a linear mapping.
On other side

$$\gamma^{\otimes n} = \sum_{k=0}^{n} S_k^n (\gamma)_k,$$

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Kernels s_k^n and S_k^n we will call Stirling kernels of first and second kind respectively.

For $f^{(n)} \in \mathcal{D}(\mathbb{R}^{nd})$

$$\left\langle (\gamma)_n, f^{(n)} \right\rangle =$$

$$\sum_{k=0}^{n} \frac{n!}{k!} \left\langle \gamma^{\otimes k}(x_1, \dots, x_k), \sum_{i_1+\dots i_k=n} \frac{(-1)^{n+k}}{i_1 \dots i_k} f^{(n)}(\underbrace{x_1, \dots, x_1}_{i_1 \text{times}}, \dots, \underbrace{x_k, \dots, x_k}_{i_k \text{times}}) \right\rangle$$

For the second kind kernels

$$\left\langle \gamma^{\otimes n}, f^{(n)} \right\rangle =$$

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Harmonic analysis on CS

Functions $G : \Gamma_0(\mathbb{R}^d) \to \mathbb{R}$ we call quasi-observables. Note that G on $\Gamma^{(n)}(\mathbb{R}^d)$ is given by a symmetric kernel $G^{(n)}(x_1, \ldots, x_n)$ and then

$$G = (G^{(n)})_{n=0}^{\infty}.$$

Functions $F : \Gamma(\mathbb{R}^d) \to \mathbb{R}$ we call observables.

For a quasi-observable G define an operator

$$(KG)(\gamma) = \sum_{\eta \subset \gamma, |\eta| < \infty} G(\eta), \ \ \gamma \in \Gamma(\mathbb{R}^d)$$

that is an observable. To be well defined we need certain assumptions about G (bounded support, ...).

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Why we call combinatorial harmonic analysis?

For $G_1, G_2 : \Gamma_0(\mathbb{R}^d) \to \mathbb{R}$ define

$$(G_1 \star G_2)(\eta) = \sum_{\eta_1 \cup \eta_2 \cup \eta_3 = \eta} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3).$$

Then

$$K(G_1 \star G_2) = KG_1 KG_2.$$

Fourier transforms of states

Let $\mu \in \mathcal{M}^1(\Gamma(\mathbb{R}^d)$. It is a state in terms of StatPhys.

$$K : Fun(\Gamma_0) \to Fun(\Gamma)$$

$$K^* : \mathcal{M}^1(\Gamma) \to \mathcal{M}(\Gamma_0).$$

$$K^*\mu = \rho, \;\; \rho = (\rho^{(n)})_{n=0}^{\infty}.$$

The measure ρ is called correlation measure for μ (Fourier transform of μ). Assume absolute continuity

$$d\rho^{(n)}(x_1, \dots, x_n) = \frac{1}{n!} k^{(n)}(x_1, \dots, x_n) dx_1 \dots x_n.$$

We call $k^{(n)}(x_1, \dots, x_n), n \in \mathbb{N}$ correlation functions of the measure μ. Transition from measures to CFs is one of the main technical aspects of the analysis on CS in applications to dynamical problems.

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Alternatively define Bogoliubov functional

$$B_\mu(\phi) = \int_{\Gamma(\mathbb{R}^d)} e^{<\gamma, \log(1+\phi>)} d\mu(\gamma).$$

Assuming B_μ is holomorphic in $\phi \in L^1(\mathbb{R}^d)$ we obtain

$$B_\mu(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int k^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) dx_1 \dots dx_n.$$

Certain related papers:

Finkelshtein, Dmitri; Kondratiev, Yuri; Lytvynov, Eugene; Oliveira, Maria Joo; Streit, Ludwig Sheffer homeomorphisms of spaces of entire functions in infinite dimensional analysis. J. Math. Anal. Appl. 479 (2019), no. 1, 162?184

Finkelshtein, Dmitri; Kondratiev, Yuri; Lytvynov, Eugene; Oliveira, Maria Joo An infinite dimensional umbral calculus. J. Funct. Anal. 276 (2019), no. 12, 3714-3766

Finkelshtein, Dmitri; Kondratiev, Yuri; Kutoviy, Oleksandr Statistical dynamics of continuous systems: perturbative and approximative approaches. Arab. J. Math. (Springer) 4 (2015), no. 4, 255-300

Marked combinatorics on $\mathbb{R}^d$

Denote $\mathbb{K}(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ the set of all discrete Radon measures on $\mathbb{R}^d$.

$$\mathbb{K}(\mathbb{R}^d) \ni \eta = \sum_i s_i \delta_{x_i}$$

where $s_i > 0$ and $\{x_i\} \subset \mathbb{R}^d$ is not (in general) a configuration.
Denote $\Pi(\mathbb{R}_+ \times \mathbb{R}^d) \subset \Gamma(\mathbb{R}_+ \times \mathbb{R}^d)$ the set of all configuration on $\mathbb{R}_+ \times \mathbb{R}^d$ with finite local mass, e.g., for $\{(s_i, x_i)\} \in \Pi(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\forall \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \ \sum_{x_i \in \Lambda} s_i < \infty.$$

Plato space

We call $\Pi(\mathbb{R}_+ \times \mathbb{R}^d)$ the Plato space w.r.t. the space $\mathbb{K}(\mathbb{R}^d)$. For $\gamma \in \Pi(\mathbb{R}_+ \times \mathbb{R}^d)$ define the reflection mapping (shine) as

$$\mathcal{R}\gamma = \sum_i s_i \delta_{x_i} \in \mathbb{K}(\mathbb{R}^d).$$

Topological and geometrical structures on $\mathbb{K}(\mathbb{R}^d)$ are obtained from corresponding objets on $\Gamma(\mathbb{R}_+ \times \mathbb{R}^d)$ using this mapping.

Definition

For each $\eta \in \mathbb{K}(\mathbb{R}^d)$, define the sequence of fake falling factorials as

$$P^{(0)}(\eta) = 1$$

$$P^{(1)}(\eta) = \eta$$

$$P^{(n)}(\eta)(x_1, \ldots, x_n) = \eta(x_1)(\eta(x_2) - s_{x_1}\delta_{x_1}(x_2))\times$$
$$\times \cdots (\eta(x_n) - s_{x_1}\delta_{x_1}(x_n) - \cdots - s_{x_{n-1}}\delta_{x_{n-1}}(x_n))$$

where $\eta = \sum_{x \in \tau(\eta)} s_x \delta_x$ and $\eta(x) = \eta(\{x\})$.

For $\eta \in \mathbb{K}(\mathbb{R}^d)$, we have

$$\frac{1}{n!}P^{(n)}(\eta) = \sum_{\{i_1,\dots,i_n\}\subset\mathbb{N}} s_{x_{i_1}} \cdots s_{x_{i_n}} \delta_{x_1} \odot \cdots \odot \delta_{x_n}$$

where $\odot$ represents the symmetric tensor product and $\eta = \sum_{i\in\mathbb{N}} s_{x_i}\delta_{x_i}$.
As stated above, the fake falling factorials also arise as the image of falling factorials on $\Pi(\mathbb{R}_+ \times \mathbb{R}^d)$ under the reflection mapping.
Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function with compact support and set $f_\varphi(s,x) := s\varphi(x)$ for $(s,x) \in$. Then for $\eta \in \mathbb{K}(\mathbb{R}^d)$ and all $n \in \mathbb{N}_0$, the following holds:

$$\langle \varphi^{\otimes n}, P^{(n)}(\eta)\rangle = \langle\langle f_\varphi^{\otimes n}, (\mathcal{R}^{-1}\eta)_n\rangle\rangle$$

where $(\cdot)_n$ denotes the falling factorials on $\Pi(\mathbb{R}_+ \times \mathbb{R}^d)$

Kondratiev, Yuri; Lytvynov, Eugene; Vershik, Anatoly Laplace operators on the cone of Radon measures. J. Funct. Anal. 269 (2015), no. 9, 2947-2976

P.Kuchling, PhD Thesis, Bielefeld, 2019

From spatial to classical combinatoric

Let $a, b : \mathbb{N} \to \mathbb{R}$. Define a convolution

$$(a \star b)(n) = \sum_{j+k+l=n} a(j+k)b(k+l).$$

As before

$$(Ka)(n) = \sum_{k+0}^{n} \binom{n}{k} a(k).$$

Then

$$K(a \star b) = Ka \cdot Kb.$$

Coherent state:

$$e_\lambda(\cdot) : \mathbb{N} \to \mathbb{C},\ e_\lambda(n) = \lambda^n, \quad \lambda \in \mathbb{C}.$$

$$(Ke_\lambda)(n) = (1+\lambda)^n.$$

Zero dimensional statistical physics

A measure $\mu \in \mathcal{M}^1(\Gamma(\mathbb{R}^d))$ is a state of a continuous system in StatPhys.
Coming back: a measure $\mu \in \mathcal{M}^1(\mathbb{N})$ is a state of $0D$ system.
Poisson measure: for $\sigma > 0$

$$\pi_\sigma(n) = e^{-\sigma} \frac{\sigma^n}{n!}.$$

Characteristics we can incorporate from the analysis on $\Gamma(\mathbb{R}^d)$:

Bogoliubov functional:

$$B(\lambda) = \int_{\mathbb{R}_+} (1+\lambda)^x d\mu(x).$$

$$(1+\lambda)^x = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (x)_n.$$

Theorem

Let $\mu \in \mathcal{M}^1(\mathbb{R}_+)$. Then $\mu(\mathbb{N}) = 1$ iff $B(\lambda)$ has a holomorphic extension.

Correlation measures

$$\int_{\mathbb{N}} (Ka)(x) d\mu(x) = \int_{\mathbb{N}} a(x) d\rho_\mu(x).$$

$$\rho_\mu(n) = \frac{1}{n!} \int_{\mathbb{N}} (x)_n d\mu(x) =$$

$$\sum_{m=n}^{\infty} \binom{m}{n} \mu(m).$$

Other $0D$ topics

Polynomial systems and new relation for them (binomial, Appel, Sheffer, Meixner)

Markov processes on N end their applications: birth-and-death, migration, infection spreading, evolutionary processes etc.

Fractional stochastic dynamics

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