Dynamics in random time

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Joint work with Jose Luis da Silva (Madeira)

Random time vs Newton time

Time-changed stochastic processes are getting increased attention due to their applications in finance, geophysics, fractional partial differential equations and in modeling the anomalous diffusion in statistical physics.

The processes that are used as time-change are generally subordinators, or inverse subordinators. Subordinators are non-decreasing Levy processes, i.e., processes with independent and stationary increments having non-decreasing sample paths.

If $S(t)$ is a subordinator, then we have

$$E[e^{-\lambda S(t)}] = e^{-t\Phi(\lambda)},$$

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Inverse subordinators

For a subordinator $S(t)$, the first-exit time process is defined by

$$E(t) = \inf_{s \geq 0} \{S(s) > t\}$$

and we call this process the inverse subordinator. Note that

$$P(E(t) > x) = P(S(x) \leq t).$$

For every jump of the subordinator $S(t)$ there is a corresponding flat period of its inverse $E(t)$. These flat periods represent trapping events in which the test particle gets immobilized in a trap. Trapping slows down the overall dynamics of the process.

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Markov case

Let $\mathcal{L}$ be a generic heuristic Markov generator defined on functions $u_0(x,t)$, $t>0$, $x\in\mathbb{R}^d$. Consider the evolution equations of the following type

$$\begin{cases} \dfrac{\partial u_0(x,t)}{\partial t} & = (\mathcal{L}u_0)(x,t) \\ u_0(x,0) & = \xi(x), \end{cases} \tag{1}$$

which we assume a solution $u_0(x,\cdot)\in L^1(\mathbb{R}_+)$ is known.

We are interested in studying the subordination of the solution $u_0(x,t)$ by the density $G_t(\tau)$ of the inverse subordinator $E(t)$, that is the function $u(x,t)$ defined by

$$u(x,t) := \int_0^\infty u_0(x,\tau)G_t(\tau)\,d\tau, \quad x\in\mathbb{R}^d,\ t\geq 0. \tag{2}$$

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If $X(t)$ be the MP for generator $\mathcal{L}$ and $Y(t) = X(E(t))$ be time changed process then

$$u_0(x,t) = E^x[\xi(X(t))]$$

$$u(x,t) = E^x[\xi(Y(t))].$$

The subordination principle tells that $u(x,t)$ is the solution of the general fractional differential equation

$$\begin{cases} (\mathbb{D}_t^{(k)} u)(x,t) & = (\mathcal{L}u)(x,t) \\ u(x,0) & = \xi(x), \end{cases} \tag{3}$$

with the same operator $\mathcal{L}$ acting in the spatial variables x and the same initial condition ξ. Here $\mathbb{D}_t^{(k)}$ is a generalized fractional (convolutional) derivative corresponding to $E(t)$ cf. Kochubei, Toaldo, ...

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Dynamical systems as Markov processes

Any DS can be considered as a deterministic MP.

Then all notions related to MP may be applied to DS.

Our aim is to demonstrate this observation.

DS and Liouville equations

Let $X(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$ be a dynamical system in $\mathbb{R}^d$ starting from x, that is, $X(0,x) = x$. This system is also a deterministic Markov process. Given $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ we define

$$u(t,x) := f(X(t,x)).$$

Then we have a version of Kolmogorov equation which is called the Liouville equation in the theory of dynamical systems:

$$\frac{\partial}{\partial t} u(t,x) = Lu(t,x),$$

where L is the generator of a semigroup.

Example

Consider DS

$$\frac{dX(t)}{dt} = V(X(t))$$

where $V \in T\mathbb{R}^d$ is a vector field.
Liouville operatror

$$Lf(x) =< V(x), \nabla f(x) >= \sum_{k=1}^{d} V_k(x) \nabla_k f(x).$$

Liouville equation

$$\frac{\partial u(t,x)}{\partial t} =< V(x), \nabla u(t,x) >$$

that is so-called transport equation.

Random times in DS

Assume we have a random time $E(t)$ as an inverse subordinator. Define a random dynamical systems

$$Y(t,x,\omega) = X(E(t,\omega),x,\omega).$$

For suitable functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ define

$$v(t,x) := E[f(Y(t,x))].$$

Then $v(t,x)$ is the solution to an evolution equation with the same generator L but with generalized fractional derivative, namely

$$\mathbb{D}_t^{(k)} v(t,x) = Lv(t,x).$$

As a result the following subordination formula holds:

$$v(t,x) = \int_0^\infty u(\tau,x) G_t(\tau)\, \mathrm{d}\tau. \tag{4}$$

The problem (as in the Markov case) is to see the change of the behavior of u after subordination.

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Example

Consider the simplest evolution equation in $\mathbb{R}^d$

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = v \in \mathbb{R}^d, \quad X(0) = x_0 \in \mathbb{R}^d.$$

The corresponding dynamics is

$$X(t) = x_0 + vt, \quad t \geq 0.$$

We consider $x_0 = 0$ just for simplicity. Assume that the certain assumptions on E are satisfied. Take $f(x) = e^{-\alpha|x|}$, $\alpha > 0$. The corresponding solution to the Liouville equation is

$$u(t, x) = e^{-\alpha t|v|}, \quad t \geq 0.$$

Using properties of densities of certain class of inverse subordinators we obtain

$$v(t,x) \sim \frac{1}{\alpha|v|\Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.$$

In particular, for the α-stable subordinator considered we obtain $v(t,x) \sim Ct^{-\alpha}$, C is a constant.

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Example

For $d = 1$ consider the dynamics

$$\alpha \frac{\mathrm{d}}{\mathrm{d}t} X(t) = \frac{1}{X^{\alpha-1}(t)}, \quad \alpha \geq 1.$$

It is clear that the solution is given by

$$X(t) = (t + C)^{1/\alpha}.$$

Take the function $f(x) = \exp(-a|x|^\alpha)$, $a > 0$, then the long time behavior of the subordination $v(t,x)$ is given by

$$v(t,x) \sim \frac{e^{-aC}}{a} \frac{t^{\gamma-1}}{\Gamma(\gamma)} Q(t), \quad t \to \infty.$$

Potentials for DS

In our framework above for a function $f : \mathbb{R}^d \to \mathbb{R}$ consider the solution to the Cauchy problem

$$\frac{\partial}{\partial t} u(t,x) = Lu(t,x),$$
$$u(0,x) = f(x).$$

Then as above

$$u(t,x) = (e^{tL} f)(x).$$

Define a potential for the function f as

$$U(f,x) = \int_0^\infty u(t,x)dt = \int_0^\infty (e^{tL} f)(x)dt = -(L^{-1} f)(x).$$

We would like to have an integral representation

$$U(f,x) = \int_{\mathbb{R}^d} f(y)\mu^x(dy)$$

with a Radon measure on $\mathbb{R}^d$. This measure we will call the Green measure for our dynamical system. The existence of this measure is a non-trivial question for each particular model.

Trajectories of random DS

Now we will analyze the transformation of trajectories of dynamical systems under random times. As above we have the Liouville equation for

$$u(t,x) := f(X(t,x)), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

that is,

$$\frac{\partial}{\partial t} u(t,x) = L u(t,x), \quad u(0,x) = f(x),$$

where L is the generator of a semigroup.

In addition, let $E(t)$, $t \geq 0$ be the inverse subordinator process, then we can consider the time changed random dynamical systems

$$Y(t,x) = X(E(t),x), \quad t \geq 0,\ x \in \mathbb{R}^d.$$

Define

$$v(t,x) := E[f(Y(t,x)].$$

The subordination formula gives

$$v(t,x) = \int_0^\infty u(\tau,x) G_t(\tau)\,\mathrm{d}\tau.$$

Now we will take the vector-function

$$f(x) = x \in \mathbb{R}^d.$$

Then the average trajectories of $Y(t,x)$ is given by

$$\int_0^\infty (X(\tau,x))G_t(\tau)\,\mathrm{d}\tau = \int_0^\infty X(\tau,x)G_t(\tau)\,\mathrm{d}\tau.$$

If we consider the dynamical system of Example 1, that is, $X(t,x) = vt$, then we obtain

$$\mathbb{E}[Y(t,x)] = v \int_0^\infty \tau G_t(\tau)\,\mathrm{d}\tau.$$

Computing the Laplace transform of each component yields

$$v\mathcal{K}(p) \int_0^\infty \tau e^{-\tau p \mathcal{K}(p)}\,\mathrm{d}\tau = \frac{vp^{-2}}{\mathcal{K}(p)} \sim vp^{-(2-\gamma)} L\left(\frac{1}{p}\right),$$

where $L(x) = (Q(x))^{-1}$ is a slowly varying function. It follows from Karamata–Feller Tauberian theorem. that

$$\mathbb{E}[Y(t,x)] \sim v \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \frac{1}{Q(t)}.$$

Diffusion approximation

Consider the following SDE in $\mathbb{R}^d$:

$$dX_\varepsilon(t) = v(X_\varepsilon)dt + \varepsilon dW(t)$$

for each $\varepsilon > 0$. The generator of this diffusion is

$$L_\varepsilon f(x) = \varepsilon \Delta f(x) + < v(x), \nabla f(x) > .$$

We can consider X_ε as small perturbation of the DS

$$dX(t) = v(X(t))dt$$

and the convergence $X_\varepsilon \to X$ under certain Lipschitz assumptions is well understood (Freidlin, Wentzel, Kutoyants,....). But the class of admissible vector fields $v(x)$ is quite restricted.

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An open problem is to have an info about transport equations

$$d/dt\ u(t,x) =< v(x), \nabla u(t,x) >$$

by means of detailed info about the heat type equations (Fokker-Planck)

$$d/dt\ u_\varepsilon(t,x) = \varepsilon \Delta u(t,x) + < v(x), \nabla u(t,x) > .$$