Random time Markov processes

Yuri Kondratiev

Bielefeld University, Germany

Center for Interdisciplinary Studies
National Dragomanov University, Kyiv

Based on joint works with Anatoly Kochubei (Kyiv)
and Jose Luis da Silva (Madeira)

Random time vs Newton time

Time-changed stochastic processes are getting increased attention due to their applications in finance, geophysics, fractional partial differential equations and in modeling the anomalous diffusion in statistical physics.

Time as a characteristic of the evolution

Biological and ecological models.

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Inverse subordinators

For a subordinator $S(t)$, the first-exit time process is defined by

$$E(t) = \inf_{s \geq 0} \{S(s) > t\}$$

and we call this process the inverse subordinator. Note that

$$P(E(t) > x) = P(S(x) \leq t).$$

For every jump of the subordinator $S(t)$ there is a corresponding flat period of its inverse $E(t)$. These flat periods represent trapping events in which the test particle gets immobilized in a trap. Trapping slows down the overall dynamics of the process.

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Markov case

Let $\mathcal{L}$ be a generic heuristic Markov generator defined on functions $u_0(x,t)$, $t>0$, $x \in \mathbb{R}^d$. Consider the evolution equations of the following type

$$\begin{cases} \dfrac{\partial u_0(x,t)}{\partial t} & = (\mathcal{L}u_0)(x,t) \\ u_0(x,0) & = \xi(x), \end{cases} \tag{1}$$

which we assume a solution $u_0(x,\cdot) \in L^1(\mathbb{R}_+)$ is known. We are interested in studying the subordination of the solution $u_0(x,t)$ by the density $G_t(\tau)$ of the inverse subordinator $E(t)$, that is the function $u(x,t)$ defined by

$$u(x,t) := \int_0^\infty u_0(x,\tau) G_t(\tau)\, d\tau, \quad x \in \mathbb{R}^d,\ t \geq 0. \tag{2}$$

If $X(t)$ be the MP for generator $\mathcal{L}$ and $Y(t) = X(E(t))$ be time changed process then

$$u_0(x,t) = E^x[\xi(X(t))]$$

$$u(x,t) = E^x[\xi(Y(t))].$$

The subordination principle tells that $u(x,t)$ is the solution of the general fractional differential equation

$$\begin{cases} (\mathbb{D}_t^{(k)} u)(x,t) & = (\mathcal{L}u)(x,t) \\ u(x,0) & = \xi(x), \end{cases} \tag{3}$$

with the same operator $\mathcal{L}$ acting in the spatial variables x and the same initial condition ξ. Here $\mathbb{D}_t^{(k)}$ is a generalized fractional (convolutional) derivative corresponding to $E(t)$ cf. Kochubei, Toaldo, ...

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in mathematics: Kolokoltsov, Leonenko, ...
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Introduction

Let $\{X_t, t \geq 0; P_x, x \in E\}$ be a strong Markov process in a phase space E. Denote T_t its transition semigroup (in a proper Banach space) and L the generator of this semigroup. Let $S_t, t \geq 0$ be a subordinator (i.e., a non-decreasing real-valued Lévy process) with $S_0 = 0$ and the Laplace exponent Φ:

$$\mathbf{E}e^{-\lambda S_t} = e^{-t\Phi(\lambda)} \;\; t, \lambda > 0.$$

We assume that S_t is independent of X_t.

Denote $E_t, t > 0$ the inverse subordinator and introduce the time changed process $Y_t = X_{E_t}$. We are interesting in the time evolution

$$u(t,x) = \mathbf{E}^x[f(Y_t)]$$

for a given initial date f. As it was pointed out in several works, $u(t,x)$ is the unique strong solution (in some proper sense) to the following Cauchy problem

$$\mathbb{D}_t^{(k)} u(t,x) = Lu(t,x) \;\; u(0,x) = f(x).$$

Here we have a generalized fractional derivative

$$\mathbb{D}_t^{(k)} \phi(t) = \frac{d}{dt} \int_0^t k(t-s)(\phi(s) - \phi(0))ds$$

with a kernel k uniquely defined by Φ.

Let $u_0(t,x)$ be the solution to a similar Cauchy problem but with ordinary time derivative. In stochastic terminology, it is the solution to the forward Kolmogorov equation corresponding to the process X_t. Under quite general assumptions there is a nice and essentially obvious relation between these evolutions:

$$u(t,x) = \int_0^\infty u_0(\tau,x) G_t(\tau) d\tau,$$

where $G_t(\tau)$ is the density of E_t. Of course, we may have similar relations for fundamental solutions to considered equations, for the backward Kolmogorov equations or time evolutions of other related quantities.

Caputo-Djrbashian fractional derivative of order $\alpha \in (0,1)$

$$\big(\mathbb{D}_t^{(\alpha)} u\big)(t) = \frac{d}{dt} \int_0^t k(t-s)\big(u(s)-u(0)\big)\, ds, \quad t>0, \tag{4}$$

where

$$k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\ t>0. \tag{5}$$

Differential-convolution operators

$$\left(\mathbb{D}_t^{(k)} u\right)(t) = \frac{d}{dt} \int_0^t k(t-s)\big(u(s) - u(0)\big)\, ds,\ t > 0, \tag{6}$$

where $k \in L^1_{\text{loc}}(\mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty)$) is a non-negative kernel.
The class of suitable kernels k we are interested in is such that the fundamental solution of the corresponding evolution equation are probability densities in $L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$.
As an example of such an operator, we consider the distributed order derivative $\mathbb{D}_t^{(\mu)}$ corresponding to

$$k(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha)\, d\alpha, \quad t > 0, \tag{7}$$

where $\mu(\alpha)$, $0 \le \alpha \le 1$ is a positive weight function on $[0, 1]$.

Differential-convolution operators

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Assumptions on the Laplace transform $\mathcal{K}$ of the kernel $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

[(H)] Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a non-negative kernel such that $\int_0^\infty k(s)\,ds > 0$ and its Laplace transform

$$\mathcal{K}(\lambda) := (Lk)(\lambda) := \int_0^\infty e^{-\lambda t} k(t)\,dt$$

exists for all $\lambda > 0$ and $\mathcal{K}$ belongs to the Stieltjes class $\mathcal{S}$ (or equivalently, the function $\mathcal{L}(\lambda) := \lambda\mathcal{K}(\lambda)$ belongs to the complete Bernstein function class $\mathcal{CBF}$ and

$$\mathcal{K}(\lambda) \to \infty, \lambda \to 0; \quad \mathcal{K}(\lambda) \to 0, \lambda \to \infty;$$

$$\mathcal{L}(\lambda) \to 0, \lambda \to 0; \quad \mathcal{L}(\lambda) \to \infty, \lambda \to \infty.$$

Concerning these classes see

R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein Functions: Theory and Applications*. De Gruyter Studies in Mathematics. De Gruyter, Berlin, 2 edition, 2012.

Example (α-Stable subordinator)

Let k be the kernel corresponding to the Caputo-Djrbashian fractional derivative $\mathbb{D}_t^{(\alpha)}$ of order $\alpha \in (0,1)$. Then its Laplace transform is given by

$$\mathcal{K}(\lambda) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-\lambda t} t^{-\alpha}\, dt = \lambda^{\alpha-1}.$$

It is easy to verify that our assumptions are satisfied for $\mathcal{K}$ and $\mathcal{L}$.

Example (Gamma subordinator)

Let k be the kernel defined by

$$\mathbb{R}_+ \ni t \mapsto k(t) := a\Gamma(0, bt), \quad a, b > 0,$$

where $\Gamma(\nu, x) := \int_x^\infty t^{\nu-1} e^{-t}\, dt$ is the upper incomplete Gamma function. The Laplace transform of k is given by

$$\mathcal{K}(\lambda) = \frac{a}{\lambda} \log\left(1 + \frac{\lambda}{b}\right), \quad \lambda > 0.$$

Again, the properties are simple to verify.

Example (Inverse Gaussian subordinator)

Let $a \geq 0$ and $b > 0$ be given and define the kernel k by

$$\mathbb{R}_+ \ni t \mapsto k(t) := \sqrt{\frac{b}{2\pi}} \left(\frac{2}{\sqrt{t}} e^{-\frac{at}{2}} - \sqrt{2a\pi}(1 - \mathsf{erf}(z)) \right), \quad z := \sqrt{\frac{at}{2}},$$

where $\mathsf{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt$ is the error function. The Laplace transform of k can be computed and is given by

$$\mathcal{K}(\lambda) = \frac{\sqrt{b}}{\lambda} \big(2\sqrt{2\lambda + a} - \sqrt{a}\big), \quad \lambda > 0.$$

The properties follow easily.

The function $[0, \infty) \ni \lambda \mapsto e^{-\tau\lambda\mathcal{K}(\lambda)}$, $\tau > 0$ is the composition of a complete Bernstein and a completely monotone function, then it is a completely monotone function. By Bernstein's theorem, for each $\tau \geq 0$, there exists a probability measure ν_τ on $\mathbb{R}_+$ such that

$$e^{-\tau\lambda\mathcal{K}(\lambda)} = \int_{(0,\infty)} e^{-\lambda s}\, d\nu_\tau(s). \tag{8}$$

Define

$$G_t(\tau) := \int_{(0,t)} k(t-s)\, d\nu_\tau(s). \tag{9}$$

Later we will see a simple probabilistic meaning of this function.

Again:

$$G_t(\tau) := \int_{(0,t)} k(t-s)\, d\nu_\tau(s). \tag{10}$$

The function $G_t(\tau)$ is a central object of our considerations.

Lemma

1. *The t-Laplace transform of $G_t(\tau)$ is given by*

$$g(\lambda,\tau) := \int_0^\infty e^{-\lambda t} G_t(\tau)\, dt = \mathcal{K}(\lambda) e^{-\tau\lambda\mathcal{K}(\lambda)}. \tag{11}$$

2. *The double (t,τ)-Laplace transform of $G_t(\tau)$ is equal to*

$$\int_0^\infty \int_0^\infty e^{-\lambda t - p\tau} G_t(\tau)\, dt\, d\tau = \frac{\mathcal{K}(\lambda)}{\lambda\mathcal{K}(\lambda)+p}.$$

3. *For each fixed $t \in \mathbb{R}_+$, $G_t(\tau)$ is a probability density, therefore $\mathbb{R}_+ \ni \tau \mapsto G_t(\tau) \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$.*

Probabilistic interpretation

Define a subordinator S by its Laplace transform as

$$\mathbf{E}(e^{-\lambda S(t)}) = e^{-t\Phi(\lambda)} = e^{-t\lambda\mathcal{K}(\lambda)}, \quad \lambda \geq 0,$$

and Φ is called the *Laplace exponent* or *cumulant* of S. The associated Lvy measure σ has support in $[0, \infty)$ and fulfills

$$\int_{(0,\infty)} (1 \wedge \tau)\, d\sigma(\tau) < \infty \tag{12}$$

and the Laplace exponent Φ can be represented by

$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda\tau})\, d\sigma(\tau). \tag{13}$$

That is known as the Lévy-Khintchine formula for the subordinator S.

The kernel k is related to the subordinator S via the Lévy measure σ, namely if we set

$$k(t) = \sigma\big((t,\infty)\big), \quad \forall t \in [0,\infty)$$

it is easy to compute its Laplace transform. In fact, for any $\lambda \geq 0$

$$\int_0^\infty e^{-\lambda t} \int_0^t d\sigma(s)\, dt = \int_0^\infty \int_0^s e^{-\lambda t}\, dt\, d\sigma(s) = \frac{1}{\lambda}\Phi(\lambda) = \mathcal{K}(\lambda).$$

Denote by E the inverse process of the subordinator S, that is

$$E(t) := \inf\{s \geq 0 : \ S(s) \geq t\} = \sup\{s \geq 0 : \ S(t) \leq s\}. \tag{14}$$

Then the marginal density of $E(t)$ is the function $G_t(\tau)$, $t, \tau \geq 0$, more precisely

$$G_t(\tau)\, d\tau = \partial_\tau \mathbf{P}(E(t) \leq \tau) = \partial_\tau \mathbf{P}(S(\tau) \geq t) = -\partial_\tau \mathbf{P}(S(\tau) < t).$$

Note that we can start with the subordinator $S(t)$ and arrive into generalized fractional derivative framework.

Fractional FPK equations

Let $\mathcal{L}$ be a generic heuristic Markov generator defined on functions $u_0(x,t)$, $t>0$, $x\in\mathbb{R}^d$. Consider the evolution equations of the following type

$$\begin{cases} \dfrac{\partial u_0(x,t)}{\partial t} & = (\mathcal{L}u_0)(x,t) \\ u_0(x,0) & = \xi(x), \end{cases} \tag{15}$$

which we assume a solution $u_0(x,\cdot)\in L^1(\mathbb{R}_+)$ is known.

We are interested in studying the subordination of the solution $u_0(x,t)$ by the density $G_t(\tau)$, that is the function $u(x,t)$ defined by

$$u(x,t) := \int_0^\infty u_0(x,\tau)G_t(\tau)\,d\tau, \quad x\in\mathbb{R}^d,\ t\geq 0. \tag{16}$$

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$$\begin{cases} (\mathbb{D}_t^{(k)}u)(x,t) & = (\mathcal{L}u)(x,t) \\ u(x,0) & = \xi(x), \end{cases} \tag{17}$$

Define the Cesaro mean

$$M_t\big(u(x,t)\big) := \frac{1}{t}\int_0^t u(x,s)\,ds$$

and investigate its long time behavior. Notice that the Cesaro mean of $u(t,x)$ may be written as

$$\begin{aligned} M_t\big(u(x,t)\big) &= \int_0^\infty u_0(x,\tau)\left(\frac{1}{t}\int_0^t G_s(\tau)\,ds\right)d\tau \\ &= \int_0^\infty u_0(x,\tau)M_t\big(G_t(\tau)\big)d\tau. \end{aligned} \tag{18}$$

Therefore we are led to investigate the Cesaro mean of the density $G_t(\tau)$ which determine the long time behavior of $u(x,t)$ once the integral in the definition exists. To this end first we introduce a suitable class of the admissible $k(t)$.

Definition (Admissible kernels - $\mathbb{K}(\mathbb{R}_+)$)

The subset $\mathbb{K}(\mathbb{R}_+) \subset L^1_{\text{loc}}(\mathbb{R}_+)$ of admissible kernels k is defined by those elements in $L^1_{\text{loc}}(\mathbb{R}_+)$ satisfying (H) such that for some $s_0 > 0$

$$\liminf_{\lambda \to 0+} \frac{1}{\mathcal{K}(\lambda)} \int_0^{\frac{s_0}{\lambda}} k(t)\, dt > 0 \tag{A1}$$

and

$$\lim_{\substack{t,r\to\infty \\ \frac{t}{r}\to 1}} \left(\int_0^t k(s)\, ds \right) \left(\int_0^r k(s)\, ds \right)^{-1} = 1. \tag{A2}$$

The assumptions (A1) and (A2) are easy to check for the classes we introduced above.

Theorem

Let $\tau \in [0, \infty)$ *be fixed and* $k \in \mathbb{K}(\mathbb{R}_+)$ *a given admissible kernel. Define the map* $G_\cdot(\tau) : [0, \infty) \longrightarrow \mathbb{R}_+$, $t \mapsto G_t(\tau)$ *such that* $\int_0^\infty e^{-\lambda t} G_t(\tau)\, dt$ *exists for all* $\lambda > 0$. *Then*

$$\lim_{\lambda \to 0+} e^{-\tau \lambda \mathcal{K}(\lambda)} = 1$$

is equivalent to

$$\lim_{t \to \infty} \left(\int_0^t G_s(\tau)\, ds \right) \left(\int_0^t k(s)\, ds \right)^{-1} = 1$$

or

$$M_t\big(G_t(\tau)\big) = \frac{1}{t} \int_0^t G_s(\tau)\, ds \sim \frac{1}{t} \int_0^t k(s)\, ds = M_t\big(k(t)\big), \quad t \to \infty$$

and $M_t\big(G_t(\tau)\big)$ *is uniformly bounded in* $\tau \in \mathbb{R}_+$.

The following three classes of admissible kernels $k \in \mathbb{K}(\mathbb{R}_+)$ are studied, and they are given in terms of their Laplace transform $\mathcal{K}(\lambda)$ as $\lambda \to 0$

$$\mathcal{K}(\lambda) \sim \lambda^{\theta-1}, \quad 0 < \theta < 1. \tag{C1}$$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(x) := \mu(0)\log(x)^{-1}. \tag{C2}$$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(x) := C\log(x)^{-1-s},\ s > 0,\ C > 0. \tag{C3}$$

(C1). We have in this case

$$\mathcal{K}(\lambda) = \lambda^{\theta-1} = \lambda^{-\rho} L\left(\frac{1}{\lambda}\right),$$

where $\rho := 1-\theta \geq 0$ and $L(x) := 1$ is a 'trivial' SVF. Applying the Karamata-Tauberian theorem we obtain

$$\int_0^t k(s)\,ds \sim Ct^{\rho}L(t) \Leftrightarrow M_t(k(t)) = \frac{1}{t}\int_0^t k(s)\,ds \sim Ct^{-\theta}, \quad \text{a}$$

(C2). We have, as $\lambda \to 0$

$$\mathcal{K}(\lambda) \sim \lambda^{-1}\left(\log\left(\frac{1}{\lambda}\right)\right)^{-1}\mu(0) = \lambda^{-1}L\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to 0,$$

where $L(x) := (\log(x))^{-1}\mu(0)$ is a SVF. Hence, by the Karamata-Tauberian theorem we obtain

$$M_t(k(t)) = \frac{1}{t}\int_0^t k(s)\,ds \sim C\log(t)^{-1}, \quad \text{as } t \to \infty.$$

(C3). The Laplace transform for each $s > 0$ as

$$\mathcal{K}(\lambda) \sim C\lambda^{-1}\left(\log\left(\frac{1}{\lambda}\right)\right)^{-1-s} = \lambda^{-1}L\left(\frac{1}{\lambda}\right), \quad \text{as } \lambda \to 0,$$

where $L(x) := C\log(x)^{-1-s}$ is a SVF. Then, by the Karamata-Tauberian theorem we obtain

$$M_t(k(t)) = \frac{1}{t}\int_0^t k(s)\,ds \sim C\log(t)^{-1-s}, \quad \text{as } t \to \infty.$$

Exponential decay

Let us assume that the solution $u_0(x,t)$ of the Cauchy problem is such that

$$\sup_{x\in\mathbb{R}^d} |u_0(x,t)| \le Ce^{-\gamma t}, \quad \gamma > 0. \tag{19}$$

As the function $\mathbb{R}_+ \ni t \mapsto u_0(x,t) \in \mathbb{R}_+$ is integrable, then the long time behavior of the Cesaro mean of

$$u(x,t) = \int_0^\infty u_0(x,\tau) G_t(\tau)\, d\tau$$

reduces to the study of the Cesaro mean of the admissible kernel $k(t)$.

(C1). For the first class of kernels (C1) it is easy to see that the Cesaro mean of k is given, as before, by

$$M_t(u(x,\cdot)) \sim Ct^{-\theta},\ t \to \infty. \tag{20}$$

(C2) For the class (C2), an application of the Karamata-Tauberian theorem gives

$$M_t(u(x,\cdot)) \sim C\log(t)^{-1},\ t \to \infty. \tag{21}$$

(C3) Now we look at class (C3) and again by the Karamata-Tauberian theorem we obtain

$$M_t(u(x,\cdot)) \sim C\log(t)^{-1-s},\ t \to \infty. \tag{22}$$

We consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u_0(x,t)}{\partial t} & = \Delta u_0(x,t) \\ u_0(x,0) & = \varphi(x), \end{cases} \tag{23}$$

where $\varphi \in L^1(\mathbb{R}^d)$. If $\mathcal{G}_t(x)$ denotes the fundamental solution then the solution $u_0(x,t)$ is written as a convolution between the initial condition φ and $\mathcal{G}_t$, that is

$$u_0(x,t) = (\varphi * \mathcal{G}_t)(x).$$

$$\sup_{x\in\mathbb{R}^d} |u_0(x,\tau)| \le C,\ \tau \in [0,1] \tag{24}$$

and

$$\sup_{x\in\mathbb{R}^d} |u_0(x,\tau)| \le \frac{C}{\tau^{d/2}},\ \tau \in]1,\infty). \tag{25}$$

The function $u(x,t)$ is defined as the subordination of $u_0(x,t)$ by the density $G_t(\tau)$, that is

$$u(x,t) = \int_0^\infty u_0(x,\tau) G_t(\tau)\, d\tau.$$

As $u_0(x,t)$ is bounded in a neighbourhood of $\tau = 0+$, then the only important contribution for the long time asymptotic of $u(x,t)$ comes from $\tau > 1$. On the other hand, the map $[1,\infty) \ni \tau \mapsto \frac{1}{\tau^{d/2}} \in \mathbb{R}_+$ belongs to $L^1(\mathbb{R}_+)$ for $d \geq 3$. Therefore we may derive the long time behavior of the Cesaro mean of $u(x,t)$ as in the previous example for each classes (C1), (C2), and (C3).

Notice that for $d = 1$ and $d = 2$ this method does not allow us to take any conclusion on the long time asymptotic of the Cesaro mean of $u(x,t)$ since $\frac{1}{\tau^{d/2}} \notin L^1(\mathbb{R}_+)$. Below we use an alternative method which allow us to do so.

The function $u(x,t)$ is defined as the subordination of $u_0(x,t)$ by the density $G_t(\tau)$, that is

$$u(x,t) = \int_0^\infty u_0(x,\tau)G_t(\tau)\,d\tau.$$

As $u_0(x,t)$ is bounded in a neighbourhood of $\tau = 0+$, then the only important contribution for the long time asymptotic of $u(x,t)$ comes from $\tau > 1$. On the other hand, the map $[1,\infty) \ni \tau \mapsto \frac{1}{\tau^{d/2}} \in \mathbb{R}_+$ belongs to $L^1(\mathbb{R}_+)$ for $d \geq 3$. Therefore we may derive the long time behavior of the Cesaro mean of $u(x,t)$ as in the previous example for each classes (C1), (C2), and (C3).
Notice that for $d = 1$ and $d = 2$ this method does not allow us to take any conclusion on the long time asymptotic of the Cesaro mean of $u(x,t)$ since $\frac{1}{\tau^{d/2}} \notin L^1(\mathbb{R}_+)$. Below we use an alternative method which allow us to do so.

Non–local Diffusion

We consider the non–local diffusion

$$\begin{cases} \frac{\partial u_0(x,t)}{\partial t} & = a * u_0(x,t) - u_0(x,t) = \int_{\mathbb{R}^d} a(x-y)(u_0(y,t)\,dy - u_0(x,t) \\ u_0(x,0) & = \varphi(x), \end{cases} \tag{26}$$

for $x \in \mathbb{R}^d$, $t > 0$, and $0 \leq a \in C(\mathbb{R}^d)$, $< a >= 1$.

Theorem

Assume that there exist $A > 0$ and $0 < r \leq 2$ such that

$$\hat{a}(\xi) = 1 - A|\xi|^r + o(|\xi|^r) \quad \text{as } \xi \to 0.$$

For any nonnegative φ such that $\varphi, \hat{\varphi} \in L^1(\mathbb{R}^d)$, there exits a unique solution $u(x,t)$ of the CP such that

$$\|u_0(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-d/r}.$$

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As the solution $u_0(x,t)$ is time continuous and uniformly bounded in x, then it is easy to derive the following properties of $u_0(x,t)$

$$\sup_{x\in\mathbb{R}^d} |u_0(x,\tau)| \le C,\ \tau\in[0,1], \tag{27}$$

$$\sup_{x\in\mathbb{R}^d} |u_0(x,\tau)| \le C\tau^{-d/r},\ \tau\in]1,\infty). \tag{28}$$

Our aim now is to study the function $u(x,t)$ given by the subordination of $u_0(x,t)$ by the density $G_t(\tau)$, namely

$$u(x,t) = \int_0^\infty u_0(x,\tau)G_t(\tau)\,d\tau,$$

that is determine the long time behavior of $u(x,t)$ for all the classes of admissible kernels $k\in\mathbb{K}(\mathbb{R}_+)$.

For $d\ge 3$ the function $\mathbb{R}_+\ni\tau\mapsto\tau^{-d/r}\in\mathbb{R}_+$ is integrable, therefore the long time behavior of $M_t(u(x,t))$ reduces to that of $M_t(G_t(\tau))$. For the three classes of admissible kernels $k\in\mathbb{K}(\mathbb{R}_+)$, they are given as above.

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Alternative Method for Subordinated Dynamics

Exponential decay

We have the exponential decay of the initial solution $u_0(x,t)$. Computing the t-Laplace transform of $u(x,t)$

$$\begin{aligned}(\mathcal{L}u(x,\cdot))(\lambda) &= C\int_0^\infty e^{-\gamma\tau}(\mathcal{L}G_\cdot(\tau))(\lambda)\,d\tau \\ &= C\mathcal{K}(\lambda)\int_0^\infty e^{-\gamma\tau}e^{-\tau\lambda\mathcal{K}(\lambda)}\,d\tau \\ &= C\frac{\mathcal{K}(\lambda)}{\lambda\mathcal{K}(\lambda)+\gamma}.\end{aligned}$$

We investigate each class of admissible kernels $k \in \mathbb{K}(\mathbb{R}_+)$, that is (C1), (C2) and (C3).

(C1). It follows that

$$(\mathcal{L}u(x,\cdot))(\lambda) = C\frac{\lambda^{\theta-1}}{\lambda^{\theta}+\gamma} = \lambda^{-(1-\theta)}L\left(\frac{1}{\lambda}\right), \quad L(x) := \frac{C}{x^{-\theta}+}$$

Then the Karamata-Tauberian theorem gives

$$M_t(u(x,t)) \sim Ct^{-\theta}\frac{1}{t^{-\theta}+\gamma} \sim Ct^{-\theta},\ t\to\infty.$$

(C2). We have, as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim C\lambda^{-1}L\left(\frac{1}{\lambda}\right), \quad L(x) := C\frac{(\log(x))^{-1}}{(\log(x))^{-1}+\gamma}.$$

And again, an application of the Karamata-Tauberian theorem yields

$$M_t(u(x,t)) \sim C\log(t)^{-1}\frac{1}{(\log(t))^{-1}+\gamma} \sim C\log(t)^{-1}, \quad t \to \infty$$

(C3). For that class one obtains

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1}L\left(\frac{1}{\lambda}\right), \quad L(x) := C\frac{(\log(x))^{-1-s}}{(\log(x))^{-1-s}+\gamma}.$$

By the Karamata-Tauberian theorem we have

$$M_t(u(x,t)) \sim C\log(t)^{-1-s}\frac{1}{(\log(t))^{-1-s}+\gamma} \sim C\log(t)^{-1-s},\ t \to \infty.$$

In conclusion, this alternative method reproduces the same type of decay of the Cesaro mean as the general method for this example.

The Heat Equation

We compute the t-Laplace transform of $u(x,t)$ and then apply the Karamata-Tauberian theorem. We have, again

$$(\mathcal{L}u(x,\cdot))(\lambda) = \mathcal{K}(\lambda)\int_0^\infty u_0(x,\tau)e^{-\tau\lambda\mathcal{K}(\lambda)}\,d\tau.$$

The long time behavior of $M_t(u(x,t))$ is only influenced as $\tau > 1$, that is the factor

$$CK(\lambda)\int_1^\infty \tau^{-d/2}e^{-\tau\lambda\mathcal{K}(\lambda)}\,d\tau.$$

The integral on the right-hand side is computed using the upper incomplete Gamma function

$$\int_b^\infty \tau^\nu e^{-\tau x}\,d\tau = x^{-\nu-1}\Gamma(\nu+1,bx), \quad \Re(x)>0. \tag{29}$$

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$$\int_b^\infty \tau^\nu e^{-\tau x}\,d\tau = x^{-\nu-1}\Gamma(\nu+1,bx), \quad \Re(x) > 0. \tag{29}$$

Hence, neglecting the constant for $\tau \in [0,1]$, the t-Laplace transform of $u(x,t)$ has the form

$$(\mathcal{L}u(x,\cdot))(\lambda) = C\mathcal{K}(\lambda)(\lambda\mathcal{K}(\lambda))^{d/2-1}\Gamma(1-d/2, \lambda\mathcal{K}(\lambda)).$$

Now we study each class of admissible kernels with the behavior described by (C1), (C2) and (C3) above.

(C1) We have $\mathcal{K}(\lambda) = \lambda^{\theta-1}$ and we distinguish the following cases:

1. For $d = 1$, as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) = C\lambda^{-(1-\theta/2)}\Gamma(1/2, \lambda^{\theta}) = \lambda^{-\rho} L\left(\frac{1}{\lambda}\right),$$

where $\rho = 1 - \theta/2$ and $L(x) := C\Gamma(1/2, x^{-\theta})$. In fact, to see that $L(x)$ is a SVF first we use the relation

$$\Gamma(s,x) = \Gamma(s) - \gamma(s,x), \quad s \neq 0, -1, -2, \ldots, \tag{30}$$

where $\gamma(s,x)$ is the lower incomplete Gamma function, the fact that $x^{-\theta} \to 0$ when $x \to \infty$ together with

$$\gamma(s,x) \sim \frac{x^s}{s}, \quad x \to 0. \tag{31}$$

Hence, by the Karamata-Tauberian theorem the Cesaro mean of $u(x,t)$ behaves as

$$M_t(u(x,t)) \sim Ct^{-\theta/2} L(t) \sim Ct^{-\theta/2},\ t \to \infty.$$

For $d = 2$, as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-(1-\theta)} L\left(\frac{1}{\lambda}\right),$$

where $L(x) := C\Gamma(0, x^{-\theta}) = CE_1(x^{-\theta})$ and $E_1(x)$, $x > 0$ is the exponential integral.
For $x \to 0$ we have

$$E_1(x) \sim -\gamma - \ln(x), \tag{32}$$

where γ is the Euler-Mascheroni constant. Then it is simple to show that $L(x) = CE_1(x^{-\theta})$ is a SVF. And again, by the Karamata-Tauberian theorem we obtain

$$M_t(u(x,t)) \sim Ct^{-\theta} L(t) \sim Ct^{-\theta}\big(\gamma + \log(t^{-\theta})\big),\ t \to \infty. \tag{33}$$

For $d \geq 3$, as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-(1-\theta)} L\left(\frac{1}{\lambda}\right),$$

where $L(x) := x^{\theta(1-d/2)}\Gamma(1-d/2, x^{-\theta})$. To show that $L(x)$ is a SVF use the relation

$$\Gamma(s,x) \sim -\frac{x^s}{s}, \quad \Re(s) < 0,\ x \to 0. \tag{34}$$

Once more, the Karamata-Tauberian theorem gives

$$M_t(u(x,t)) \sim Ct^{-\theta}L(t) \sim Ct^{-\theta},\ t \to \infty.$$

(C2) The Laplace transform $\mathcal{K}(\lambda)$ behaves as $\lambda \to 0$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \qquad L(x) := \mu(0)\log(x)^{-1}.$$

We distinguish the cases $d = 1$, $d = 2$ and $d \geq 3$.

1. For $d = 1$ as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where

$$L(x) := C\log(x)^{-1/2}\Gamma(1/2, \mu(0)\log(x)^{-1}).$$

$L(x)$ is a SVF. Hence, by the Karamata-Tauberian theorem the Cesaro mean of $u(x,t)$ is

$$M_t(u(x,t)) \sim C\log(t)^{-1}\left(\log(t)^{1/2}\Gamma(1/2, \mu(0)\log(t)^{-1})\right)$$
$$\sim C\log(t)^{-1} + C'\log(t)^{-1/2},\ t \to \infty.$$

For $d = 2$ as $\lambda \to 0$

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) := \mu(0)\log(x)^{-1} E_1(\mu(0)\log(x)^{-1})$. Again, $L(x)$ is a SVF because it is the product of two SVF. Then an application of the Karamata-Tauberian theorem yields

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1} E_1(\mu(0)\log(t)^{-1}) \\ &\sim C\log(t)^{-1}\left[\gamma + \log\left(\mu(0)\log(t)^{-1}\right)\right],\ t \to \infty. \end{aligned}$$

In general, for any $d \geq 3$ as $\lambda \to 0$ we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where

$$L(x) := \left(\mu(0)\log(x)^{-1}\right)^{d/2} \Gamma(1 - d/2, \mu(0)\log(x)^{-1}).$$

It is clear that $L(x)$ is a SVF, hence the Karamata-Tauberian theorem implies the long time behavior for $M_t(u(x,\cdot))$, namely

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1}\left(\log(t)^{1-d/2}\Gamma(1 - d/2, \mu(0)\log(t)^{-1})\right. \\ &\sim C\log(t)^{-1}. \end{aligned}$$

(C3) Finally, let us investigate the Cesaro mean of $u(x,t)$ for the class (C3), that is where $\mathcal{K}(\lambda)$ behaves as $\lambda \to 0$

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(x) := C(\log(x))^{-1-s},\ s > 0,\ C > 0.$$

Proceeding as before we distinguish the following cases:
For $d = 1$, as $\lambda \to 0$ we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where

$$\begin{aligned} L(x) &:= C\log(t)^{-1-s}\Gamma(1/2, C\log(x)^{-1-s}) \\ &= C\log(t)^{-1-s}\big(\sqrt{\pi} - \gamma(1/2, \log(x)^{-1-s}\big) \end{aligned}$$

is a SVF since it is the product of two SVF. Then, the Karamata-Tauberian theorem yields

$$M_t(u(x,t)) \sim C\log(t)^{-1-s}\left(\sqrt{\pi} - 2\log(t)^{(-1-s)/2}\right),\ t \to \infty.$$

For $d = 2$, as $\lambda \to 0$ we have

As a conclusion, the alternative method produces the same long time decay of the Cesaro mean of $u(x,t)$ compared to the general method. In addition, with the Laplace transform method we can handle the dimensions $d = 1$ and $d = 2$ which was not possible with the general method.

Non-local Diffusion

The long time behavior of $M_t(u(x,t))$ depends only on $\tau > 1$, that is the factor

$$CK(\lambda)\int_1^\infty \tau^{-d/r}e^{-\tau\lambda\mathcal{K}(\lambda)}\,d\tau.$$

The integral on the right hand side above is (neglecting a constant)

$$(\mathcal{L}u(x,\cdot))(\lambda) = C\mathcal{K}(\lambda)(\lambda\mathcal{K}(\lambda))^{d/r-1}\Gamma(1-d/r,\lambda\mathcal{K}(\lambda)).$$

We investigate the long time behavior of $M_t(u(x,t))$ for the three classes of admissible kernels (C1) , (C2) and (C3). The analysis below is similar to the analysis of the heat equation assuming $1 < r \leq 2$.

(C1) We have $\mathcal{K}(\lambda) = \lambda^{\theta-1}$, $0 < \theta < 1$ and

$$(\mathcal{L}u(x,\cdot))(\lambda) = \lambda^{-(1-\theta d/r)} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\Gamma(1-d/r, x^{-\theta})$ which is a SVF.

For $d=1$ it follows that

$$(\mathcal{L}u(x,\cdot))(\lambda) = \lambda^{-(1-\theta/r)}\Gamma(1-1/s, \lambda^{-\theta})$$

with $1-\theta/r > 0$ and $1-1/r \in (0, 1/2]$. As $\Gamma(1-1/r, \lambda^{-\theta})$ is a SVF, then the Karamata-Tauberian theorem gives

$$M_t(u(x,t)) \sim C\lambda^{-\theta/r}\Gamma(1-1/r, \lambda^{-\theta})$$

and

$$M_t(u(x,t)) \sim Ct^{-\theta/r}L(t) \sim Ct^{-\theta/r}.$$

For $d = 2$ we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \geq \lambda^{-(1-2\theta/r)}\Gamma(1-2/r, \lambda^{-\theta})$$

such that to have $1 - 2\theta/r > 0$ implies that $r = 2$. This case is similar to the heat equation. Thus, we have

$$M_t(u(x,t)) \sim Ct^{-\theta}\big(\gamma + \log(t^{-\theta})\big), \quad t \to \infty.$$

For $d \in [3, r/\theta \vee 3)$, we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \geq \lambda^{-(1-\theta)} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = x^{\theta(1-d/r)}\Gamma(1-d/r, x^{-\theta})$ is a SVF. Therefore, we derive the long time behavior of $M_t(u(x,t))$ as a consequence of the Karamata-Tauberian theorem, namely

$$M_t(u(x,t)) \sim Ct^{-\theta}L(t) \sim Ct^{-\theta}, \quad t \to \infty.$$

(C2) That is the case when

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(x) := \mu(0)\log(x)^{-1}$$

which implies

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim C\lambda^{-1} L\left(\frac{1}{\lambda}\right)^{d/r} \Gamma\left(1-d/r, L\left(\frac{1}{\lambda}\right)\right).$$

For $d=1$ as $\lambda \to 0$, we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-1/r}\Gamma(1-1/r, \mu(0)\log(x)^{-1})$ is a SVF. Then the Karamata-Tauberian theorem yields

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1}\log(t)^{-1-1/r}\Gamma(1-1/r, \mu(0)\log(x)^{-1}) \\ &\sim C\log(t)^{-1}\big(\Gamma(1-1/r)\log(t)^{-1-1/r} - C'\big), \quad t \to \end{aligned}$$

Now for $d = 2$ we have

$$\mathcal{L}(u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-2/r}\Gamma(1-2/r, \mu(0)\log(x)^{-1})$ is a SVF.

For the special case $r = 2$ it reduces to

$$L(x) = C\log(x)^{-1}E_1(\mu(0)\log(x)^{-1}).$$

Then an application of the Karamata-Tauberian theorem yields

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1}E_1(\mu(0)\log(t)^{-1}) \\ &\sim C\log(t)^{-1}\big[\gamma + \log(\mu(0)\log(t)^{-1})\big], \quad \text{t}\to\infty. \end{aligned}$$

For $1 < r < 2$, then $-1 < 1 - 2/r < 0$ and

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1}\log(x)^{1-2/r}\Gamma(1-2/r, \mu(0)\log(x)^{-1}) \\ &\sim C\log(t)^{-1}, \quad \text{t}\to\infty. \end{aligned}$$

For $d \geq 3$, we obtain

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-d/r}\Gamma(1-d/r, \mu(0)\log(x)^{-1})$ is a SVF. As $1-d/r<0$, then by the Karamata-Tauberian theorem follows

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1}\log(x)^{1-d/r}\Gamma(1-d/r, \mu(0)\log(x)^{-1}) \\ &\sim C\log(t)^{-1}, \quad \text{t} \to \infty. \end{aligned}$$

(C3) The third class of admissible kernels has Laplace transform

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right), \quad L(x) := C(\log(x))^{-1-s},\ s>0,\ C>0$$

such that

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim C\lambda^{-1} L\left(\frac{1}{\lambda}\right)^{d/r} \Gamma\left(1-d/r, L\left(\frac{1}{\lambda}\right)\right).$$

First we take $d=1$ and obtain

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-(1+s)/r}\Gamma(1-1/r, C\log(x)^{-1-s})$ is a SVG. Then by the Karamata-Tauberian theorem and follows

$$M_t(u(x,t)) \sim C\log(t)^{-1-s}\log(t)^{1+s-(1+s)/r}\Gamma(1-1/r, C\log(t$$

$$\sim C\log(t)^{-1-s}\left(\log(t)^{1+s-(1+s)/r}\Gamma(1-1/r) + C'\right.$$

For $d = 2$ we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-2(1+s)/r}\Gamma\left(1 - 2/r, C\log(x)^{-1-s}\right)$ is a SVF.

For $r = 2$ the SVF $L(x)$ reduces to

$$L(x) = C\log(x)^{-(1+s)}E_1\left(C\log(x)^{-1-s}\right)$$

and then we obtain

$$M_t(u(x,t)) \sim C\log(t)^{-1-s}(\gamma + \log(C\log(t)^{-1-s})), \quad t \to \infty.$$

For $1 < r < 2$ then $-1 < 1 - 2/r < 0$ and

$$M_t(u(x,t)) \sim C\log(t)^{-1-s}\log(t)^{(1+s)(1-2/r)}\Gamma\left(1 - 2/r, C\log(t\right.$$

$$\sim C\log(t)^{-1-s}, \quad t \to \infty.$$

Finally for $d \geq 3$ we have

$$(\mathcal{L}u(x,\cdot))(\lambda) \sim \lambda^{-1} L\left(\frac{1}{\lambda}\right),$$

where $L(x) = C\log(x)^{-d(1+s)/r}\Gamma\left(1 - d/r, C\log(x)^{-1-s}\right)$ is a SVF. As before, we obtain

$$\begin{aligned} M_t(u(x,t)) &\sim C\log(t)^{-1-s}\log(t)^{(1+s)(1-d/r)}\Gamma\left(1 - d/r, C\log(t \right. \\ &\sim C\log(t)^{-1-s}, \quad t \to \infty. \end{aligned}$$

In conclusion, both methods produces the same type of long time behavior for $d \geq 3$, in addition for $d = 1$ and $d = 2$ we are also able to obtain a decay using this alternative Laplace transform method.

Asymptotic without Cesaro mean

Classical relaxation equation

$$u'(t) = -\lambda u(t), \;\; u(0) = 1, \lambda > 0.$$

Fractional relaxation equation

$$\mathbb{D}_t^{(k)} v(t) = -\lambda v(t).$$

In particular, we have models with decaying correlation functions:

$$\varkappa_t^{(1)} = e^{-\lambda t}, \quad \beta > 0. \tag{35}$$

This situation is realized in the contact model in subcritical regime.

Theorem

Assume

$$\mathcal{K}(p) \sim p^{-\gamma} Q\left(\frac{1}{p}\right), \quad p \to 0, \tag{36}$$

where $0 \leq \gamma \leq 1$, Q is a slowly varying function.

Then for the solution $v(t)$ holds

$$v(t) \sim \frac{1}{\Gamma(\gamma)\lambda} t^{\gamma-1} Q(t), \quad t \to \infty.$$

Examples

1) In the case of the Caputo-Djrbashian fractional derivative $\mathbb{D}^\alpha$, $0 < \alpha < 1$, we have $\mathcal{K}(p) = p^{\alpha-1}$, and (36) is satisfied.
2) For the distributed order derivative with a continuous weight function μ, we have

$$\mathcal{K}(p) \sim p^{-1} \left(\log \frac{1}{p} \right)^{-1} \mu(0), \quad p \to 0,$$

if $\mu(0) \neq 0$. Thus, in this case (36) holds with $\gamma = 1$.

Relaxation process equation

$$\frac{du}{dt} = -\lambda(u(t) - u_0),$$

where $1/\lambda$ is called the relaxation time. The solution

$$u(t) = u_0(1 - e^{-\lambda t})$$

is growing to the equilibrium state u_0. In the fractal dynamics we will have

$$u_0(1 - v(t)).$$

That means that the convergence to the equilibrium value u_0 will be essentially different from the classical case.

Thanks for your attention!