NOTES ON PRINCIPAL BUNDLES, CONNECTIONS, AND HOMOGENEOUS SPACES

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1. Introduction

These notes are meant to accompany talks in the Geometry Seminar at Texas Tech during the Spring Semester of 2009.

These notes will be updated frequently, so it's best to keep track of them on the web. Note the version time stamp at the bottom of the first page.

At this point, the seminar participants should be comfortable with the basic differential geometry apparatus done from the point of view of principal bundles.

I hope to come back and add this material to these notes at some point in the future.

Thanks to the seminar participants for listening to me and straightening me out. Any errors are, of course, my own.

2. Group Actions

Let G be a Lie Group. If $g \in G$ we have the multiplication maps

$$L_g \colon G \to G \colon x \mapsto gx$$
$$R_g \colon G \to G \colon x \mapsto xg.$$

Note that for $a, b \in G$, we have $L_a \circ R_b = R_b \circ L_a$, by the associative law.

We'll use e to denote the identity element of G.

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We identify $\mathfrak{g}$, the Lie Algebra of G, with T_eG, the tangent space at the identity. If $A \in \mathfrak{g}$, we denote the left invariant vector field generated by A by $\lambda(A)$, and we denote the value of this vector field at $g \in G$ by $\lambda_g(A)$. Thus, we have

$$\lambda_g(A) = (L_g)_* A \in T_gG. \tag{2.1}$$

where

$$(L_g)_* \colon T_eG \to T_gG$$

is the tangent map of L_g.

Notation 2.1. In what follows we will usually drop the lower star when writing the tangent map of L_g or R_g. Thus, we will usually write the equation (2.1) as just

$$\lambda_g(A) = L_g A.$$

This simplifies the notation and there is usually no danger of confusion, since there is not much else that $L_g A$ could mean.

The Lie bracket of two left invariant vector fields is left invariant, and we define the bracket on $\mathfrak{g}$ by

$$[A, B] := [\lambda(A), \lambda(A)]_e.$$

Thus, in general,

$$[\lambda(A), \lambda(B)] = \lambda([A, B])$$

We denote the right invariant vector field generated by $A \in \mathfrak{g}$ by $\rho(A)$. Since we used left invariant vector fields to define the bracket in $\mathfrak{g}$, we have for right invariant vector fields

$$[\rho(A), \rho(B)]_e = -[A, B]$$

and so

$$[\rho(A), \rho(B)] = -\rho([A, B]).$$

For $A \in \mathfrak{g}$ we write $\exp(tA) = e^{tA}$ for the exponential map of G. We have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} R_{e^{tA}}(g) &= \left.\frac{d}{dt}\right|_{t=0} g e^{tA} \\ &= \left.\frac{d}{dt}\right|_{0} L_g(e^{tA}) \\ &= L_g A \\ &= \lambda_g(A). \end{aligned}$$

Thus the flow $\{\varphi_t\}$ of $\lambda(A)$ is $\varphi_t = R_{e^{tA}}$. Similarly, the flow $\{\psi_t\}$ of $\rho(A)$ is $\psi_t = L_{e^{tA}}$.

The following observations will be useful in a moment. Let M and N be manifolds. Let $\pi_1 \colon M \times N \to M$ and $\pi_2 \colon M \times N \to N$ be the projections. Recall that one usually identifies $T_{(p,q)}(M \times N)$ with $T_p M \times T_q N$ via the map

$$T_{(p,q)}(M \times N) \ni X \mapsto ((\pi_1)_* X, (\pi_2)_* X) \in T_p M \times T_q N.$$

Given a vector field X on M, we can define a vector field $\hat{X}$ on $M \times N$ by $\hat{X}_{(p,q)} = (X_p, 0_q)$. We can characterize $\hat{X}$ as the unique vector field that is π_1-related to X and π_2-related to the zero vector field.

If X_1 and X_2 are vector fields on M, we see that $[\hat{X}_1, \hat{X}_2]$ is π_1-related to $[X_1, X_2]$ and π_2-related to $[0, 0] = 0$. Thus, $[\hat{X}_1, \hat{X}_2] = [X_1, X_2]\widehat{\ }$

Now suppose that G acts on M on the left. Let A be an element of the Lie Algebra $\mathfrak{g}$. We can define a flow $\{\varphi_t\}$ on M by

$$\varphi_t(p) = e^{tA} p.$$

This flow comes from a vector field, which we will denote by $\rho^M(A)$. Thus,

$$\rho^M_p(A) = \left.\frac{d}{dt}\right|_{0} e^{tA} p.$$

We now have a map $A \mapsto \rho^M(A)$ from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Naturally, we want to know what $[\rho^M(A), \rho^M(B)]$ is. The answer is

$$[\rho^M(A), \rho^M(B)] = -\rho^M([A, B]).$$

To see this, let

$$m \colon G \times M \to M \colon (g, p) \mapsto gp$$

be the group action. Given $A \in \mathfrak{g}$, we have the right invariant vector field $\rho(A)$ on G, and so we get the vector field $\hat{\rho}(A)$ on $G \times M$ as above. We have

$$\begin{aligned} m_*(\hat{\rho}_{(g,p)}(A)) &= m_*(\rho_g(A), 0_p) \\ &= \frac{d}{dt}\bigg|_0 m(e^{tA}g, p) \\ &= \frac{d}{dt}\bigg|_0 e^{tA}gp \\ &= \rho^M_{gp}(A). \end{aligned}$$

Thus, $\rho^M(A)$ is the unique vector field on M that is m-related to $\hat{\rho}(A)$. If B is also in $\mathfrak{g}$, then $[\hat{\rho}(A), \hat{\rho}(B)]$ is m-related to $[\rho^M(A), \rho^M(B)]$. But

$$[\hat{\rho}(A), \hat{\rho}(B)] = [\rho(A), \rho(B)]\widehat{\ } = -\hat{\rho}([A, B]).$$

Next, we consider the action on these vector fields of equivariant maps.

Proposition 2.2. *Let G act on the left of manifolds M and N and let $f\colon M \to N$ be an equivariant map, i.e., $f(gp) = gf(p)$. The we have*

$$f_*(\rho^M_p(A)) = \rho^N_{f(p)}(A), \qquad A \in \mathfrak{g}.$$

In other words, $\rho^M(A)$ is f-related to $\rho^N(A)$.

Proof. This is a simple calculation:

$$\begin{aligned} f_*(\rho^M_p(A)) &= \frac{d}{dt}\bigg|_0 f(e^{tA}p) \\ &= \frac{d}{dt}\bigg|_0 e^{tA}f(p) \\ &= \rho^N_{f(p)}(A). \end{aligned}$$

□

We can also see the effect of a group translation on $\rho^M(A)$.

Proposition 2.3. *Let G act on M on the left. Then, for $g \in G$ and $A \in \mathfrak{g}$, we have*

$$L_g \rho^M_x(A) = \rho^M_{gx}(\mathrm{Ad}(g)A),$$

where Ad is the adjoint representation of G.

Proof. Just calculate:

$$\begin{aligned} L_g\rho^M_x(A) &= \frac{d}{dt}\bigg|_0 L_g(e^{tA}x) \\ &= \frac{d}{dt}\bigg|_0 ge^{tA}x \\ &= \frac{d}{dt}\bigg|_0 ge^{tA}g^{-1}gx \\ &= \frac{d}{dt}\bigg|_0 e^{t\,\mathrm{Ad}(g)A}gx \\ &= \rho^M_{gx}(\mathrm{Ad}(g)A). \end{aligned}$$

□

For a fixed $p \in M$, we have the map $\rho^M_p(\cdot)\colon \mathfrak{g} \to T_pM$. We want to know something about the kernel and the image of this map.

For p in M, let H_p be the isotropy group of p, i.e.,

$$H_p := \{g \in G \mid gp = p\}.$$

This is a closed subgroup of G. The Lie algebra of H_p will be denoted by $\mathfrak{h}_p$.

Proposition 2.4. *The kernel of $\rho^M_p(\cdot)\colon \mathfrak{g} \to T_pM$ is $\mathfrak{h}_p$.*

Proof. If $B \in \mathfrak{h}_p$, then $e^{tB} \in H_p$ of all t. Thus,

$$e^{tB}p = p. \tag{2.2}$$

Differentiating this equation at $t = 0$ gives $\rho^M_p(B) = 0$.

Conversely, suppose that $\rho^M_p(B) = 0$. This means that p is a zero of the vector field $\rho^M(B)$, as so p is an equilibrium point of the flow. This means that (2.2) holds for all t. Thus, $e^{tB} \in H_p$ for all t, and this implies that $B \in \mathfrak{h}_p$.

□

The main result we need about the image of this map is the following.

Proposition 2.5. *If G act transitively on M, the map $\rho^M_p(\cdot)\colon \mathfrak{g} \to T_pM$ is surjective.*

This follows from an important theorem discussed earlier in the seminar.

Theorem 2.6. *Let G act transitively on M. Fix $p \in M$ and define $\pi_G \colon G \to M \colon g \mapsto gp$. Then $\pi_G \colon G \to M$ has the structure of a principal H_p bundle. In particular, π_G is a submersion.*

To get Proposition 2.5, note that the last theorem implies that $(\pi_G)_* \colon \mathfrak{g} \to T_pM$ is surjective. But

$$\begin{aligned}
(\pi_G)_*(A) &= \frac{d}{dt}\bigg|_0 \pi_G(e^{tA}) \\
&= \frac{d}{dt}\bigg|_0 e^{tA}p \\
&= \rho_p^M(A).
\end{aligned}$$

Remark 2.7. In general, $\rho_p^M(\mathfrak{g})$ is the tangent space to the orbit of p, which is an immersed submanifold.

We can, of course, repeat these constructions with right actions. We merely record the results.

Proposition 2.8. *Suppose that G acts on the right of M.*

(1) *For each $A \in \mathfrak{g}$, we get a vector field $\lambda^M(A)$ on M by*

$$\lambda_p^M(A) = \frac{d}{dt}\bigg|_0 pe^{tA}$$

(2) *We have*

$$[\lambda^M(A), \lambda^M(A)] = \lambda^M([A, B])$$

(3) *We have*

$$R_g \lambda_p^M(A) = \lambda_{pg}^M(\mathrm{Ad}(g^{-1})A).$$

The results about the kernel and image are the same.

3. General Invariant Connections

In this section we study invariant connections on a homogeneous space, without assuming any special structure on the homogeneous space. Later we will consider the reductive case.

3.1. **The Basic Setup.** We begin with some notation.

Notation 3.1. If V is a vector space, $\mathrm{GL}(V)$ denotes the Lie Group of linear automorphisms of G. The Lie algebra of $\mathrm{GL}(V)$ will be identified, in the usual way, with $\mathfrak{gl}(V)$, the space of linear endomorphisms of V, with the commutator bracket.

We want to study homogeneous spaces, so we assume the following setup.

Let M be a manifold and assume that a Lie group G acts **transitively** on the left of M. Fix a base point p_0 in M, and let $H = H_{p_0}$ be the isotropy group at p_0.

We choose a model space V for the tangent spaces of M and we fix a particular isomorphism

$$u_0 \colon V \to T_{p_0}M.$$

Later on we will have natural choices for V and u_0, but we won't specify them yet.

If $h \in H$, we have $L_h p_0 = p_0$, so we get an induced map

$$T_{p_0}M \xrightarrow{T_{p_o}L_h} T_{p_o}M.$$

The mapping $H \to \mathrm{GL}(T_{p_0}M) \colon h \mapsto T_{p_0}L_h$ is a representation of H called the **Isotropy Representation**.

Since u_0 is an isomorphism, we get an induced map $\alpha(h) \colon V \to V$ so that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{u_0} & T_{p_0}M \\ {\scriptstyle \alpha(h)}\downarrow & & \downarrow{\scriptstyle T_{p_0}L_h} \\ V & \xrightarrow[u_0]{} & T_{p_0}M \end{array} \tag{3.1}$$

Thus, we get a representation $\alpha \colon H \to \mathrm{GL}(V)$, which is equivalent to the isotropy representation. Just to have a name for it, let's call α the **Model Representation**.

We now want to form the bundle of V-frames of M. A V-frame at $p \in M$ is, by definition, a linear isomorphism

$$u \colon V \to T_pM.$$

The set of V-frames at p is denoted by $F_p(M)$. The group $\mathrm{GL}(V)$ acts transitively and freely on the right of $F_p(M)$ by composition, i.e. $ua := u \circ a$, as in the following commutative diagram.

$$\begin{array}{ccc} V & \xrightarrow{\ u\ } & T_pM \\ {\scriptstyle a}\uparrow & \nearrow {\scriptstyle ua} & \\ V & & \end{array}$$

We set $F(M) = \bigcup_p F_p(M)$ and define a projection $\pi\colon F(M) \to M$ by $\pi(F_p(M)) = \{p\}$. As discussed in the seminar,

$$\begin{array}{c} F(M) \\ \downarrow {\scriptstyle \pi} \\ M \end{array}$$

has the structure of a $\mathrm{GL}(V)$-principal bundle.

Recall that the vertical space $\mathscr{V}_u \subseteq T_uF(M)$ is the tangent space to the fiber. It can also be described as the kernel of $T_u\pi$. The group $\mathrm{GL}(V)$ acts on the right of $F(M)$ and the orbits are exactly the fibers. As in our discussion of group actions, an element $A \in \mathfrak{gl}(V)$ gives us a vertical vector field

$$\mathscr{V}_u(A) = \frac{d}{dt}\bigg|_0 ue^{tA}.$$

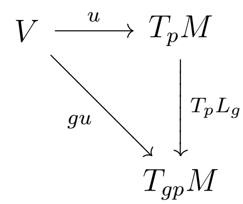
We used $\lambda_u^{F(M)}(A)$ for this in the last section, but we'll use $\mathscr{V}_u(A)$ in this case to remind us that it's vertical. Since this vector field comes from a right action, we have

$$R_a\mathscr{V}_u(A) = \mathscr{V}_{ua}(\mathrm{Ad}(a^{-1})A). \tag{3.2}$$

Since the action of $\mathrm{GL}(V)$ is free (i.e., no isotropy) the map $A \mapsto \mathscr{V}_u(A)$ is an isomorphism $\mathfrak{gl}(V) \to \mathscr{V}_u$ (to see that it's surjective, use a local trivialization).

Since G acts on the left of M, we can define a left action of G on $F(M)$. Suppose $g \in G$; since L_g is a diffeomorphism of M, it induces an isomorphism from T_pM to $T_{gp}M$. If $u \in F_p(M)$, we define

$gu = L_g \circ u$, as in the following commutative diagram.



Note that $\pi(gu) = g\pi(u)$, i.e., π is an equivariant map.

We now have two group actions on $F(M)$, one on the left and one on the right. Note that for $u \in F(M)$, $g \in G$ and $a \in \mathrm{GL}(V)$ we have

$$(gu)a = g(ua),$$

just because composition of maps is associative. This fact can be written as

$$R_a L_g = L_g R_a, \qquad g \in G,\ a \in \mathrm{GL}(V), \tag{3.3}$$

so we can say that the actions commute.

Since G acts on the left of M, every $x \in \mathfrak{g}$ gives us a vector field $\rho^M(x)$ on M. Since G acts on the left of $F(M)$, we have a similar vector field $\rho^{F(M)}(x)$. Since π is an equivariant map,

$$(\pi)_*\Big(\rho_u^{F(M)}(x)\Big) = \rho^M_{\pi(u)}(x).$$

Notation 3.2. Usually we'll drop the superscripts on the vector fields $\rho^M(x)$, etc. It should be clear from context where the vector field lives.

Since the vector fields $\rho(x)$ come from a left action, we have

$$L_g \rho_p(x) = \rho_{gp}(\mathrm{Ad}(g)x)$$

for $p \in M$, and a similar equation on $F(M)$.

We can interpret (3.3) as saying that R_a is an equivariant map with respect to the action of G, so

$$R_a \rho_u(x) = \rho_{ua}(x).$$

Similarly, we can interpret (3.3) as saying that L_g is equivariant with respect to the action of $\mathrm{GL}(V)$, so

$$L_g \mathscr{V}_u(A) = \mathscr{V}_{gu}(A). \tag{3.4}$$

Note that in the diagram(3.1), the composition $T_{p_0}L_h \circ u_0$ is what we've defined to be hu_0, and that the composition $u_0 \circ \alpha(h)$ is the right action of $\alpha(h) \in \mathrm{GL}(V)$ on u_0. Thus, we have the formula

$$\boxed{hu_0 = u_0\alpha(h), \qquad h \in H.} \tag{3.5}$$

We've boxed this formula because we will use it frequently.

Here is the infinitesimal version.

Proposition 3.3. *Suppose that $x \in \mathfrak{g}$. Then $\rho_{u_0}(x)$ is vertical if and only if $x \in \mathfrak{h}$. In this case,*

$$\rho_{u_0}(x) = \mathscr{V}_{u_0}(\alpha_*(x)),$$

where $\alpha_* \colon \mathfrak{h} \to \mathfrak{gl}(V)$ is the Lie algebra homomorphism induced by the model representation $\alpha \colon H \to \mathrm{GL}(V)$.

Proof. We know that $\rho_{u_0}(x)$ is vertical if and only if $\pi_*\rho_{u_0}(x) = 0$. But $\pi_*\rho_{u_0}(x) = \rho_{p_0}(x)$ and the kernel of $\rho_{p_0}(\cdot)$ is $\mathfrak{h}$.

If $x \in \mathfrak{h}$, we have

$$\begin{aligned}\rho_{u_0}(x) &= \left.\frac{d}{dt}\right|_0 e^{tx}u_0 \\ &= \left.\frac{d}{dt}\right|_0 u_0\alpha(e^{tx}) \\ &= \left.\frac{d}{dt}\right|_0 u_0e^{t\alpha_*(x)} \\ &= \mathscr{V}_{u_0}(\alpha_*(x)).\end{aligned}$$

□

This completes the description of the basic setting we will use. We next consider invariant connections.

3.2. Invariant Connections.

Definition 3.4. Recall that a connection on $F(M)$ can be specified as a distribution $\mathscr{H}$ on $F(M)$ such that

(1) $TF(M) = \mathscr{H} \oplus \mathscr{V}$
(2) $R_a\mathscr{H}_u = \mathscr{H}_{ua}$ for all $u \in F(M)$ and $a \in \mathrm{GL}(V)$.

There are several ways to define the notion of a connection being invariant under G, i.e., G acts on M by affine maps. It can be described in terms of parallel translation and covariant derivatives, but the following definition should be plausible.

Definition 3.5. We say that $\mathscr{H}$ is **G-invariant** if the action of G preserves $\mathscr{H}$, i.e.

$$L_g \mathscr{H}_u = \mathscr{H}_{gu}, \qquad \forall g \in G,\ u \in F(M).$$

Now, recall (3.5). If $h \in H$, then $hu_0 = u_0\alpha(h)$, so $R_{\alpha(h^{-1})} L_h u_0 = u_0$. Thus, we get an induced map

$$\psi(h) := R_{\alpha(h^{-1})} L_h \colon T_{u_0}F(M) \to T_{u_0}F(M) \tag{3.6}$$

Lemma 3.6. *The map*

$$\psi\colon H \to \mathrm{GL}(T_{u_0}F(M))$$

is a representation (i.e., a group homomorphism).

Proof of Lemma. It's clear that $\psi(e)$ is the identity. Suppose that h and k are in H. Then

$$\begin{aligned}
\psi(hk) &= R_{\alpha((hk)^{-1})} L_{hk} \\
&= R_{\alpha(k^{-1}h^{-1})} L_{hk} \\
&= R_{\alpha(k^{-1})\alpha(h^{-1})} L_{hk} \\
&= R_{\alpha(h^{-1})} R_{\alpha(k^{-1})} L_h L_k \\
&= R_{\alpha(h^{-1})} L_h R_{\alpha(k^{-1})} L_k \qquad \text{by (3.3)}, \\
&= \psi(h)\psi(k).
\end{aligned}$$

□

Lemma 3.7. *The vectical space $\mathscr{V}_{u_0}$ is invariant under ψ.*

Proof of Lemma. We have

$$\begin{aligned}
\psi(h)\mathscr{V}_{u_0}(A) &= R_{\alpha(h^{-1})} L_h \mathscr{V}_{u_0}(A) \\
&= R_{\alpha(h^{-1})} \mathscr{V}_{hu_0}(A) \qquad \text{by (3.4)}, \\
&= R_{\alpha(h^{-1})} \mathscr{V}_{u_0\alpha(h)}(A) \\
&= \mathscr{V}_{u_0}(\mathrm{Ad}(\alpha(h))A) \qquad \text{by (3.2)}.
\end{aligned}$$

□

Suppose now that $\mathscr{H}$ is an invariant connection. We then have

$$\begin{aligned}\psi(h)\mathscr{H}_{u_0} &= R_{\alpha(h^{-1})}L_h\mathscr{H}_{u_0}\\ &= R_{\alpha(h^{-1})}\mathscr{H}_{hu_0} \qquad (\mathscr{H} \text{ is invariant}),\\ &= R_{\alpha(h^{-1})}\mathscr{H}_{u_0\alpha(h)}\\ &= \mathscr{H}_{u_0}.\end{aligned}$$

We can go the other way, and construct an invariant connection from a ψ-invariant horizontal space at u_0.

Theorem 3.8. *There is a one-to-one correspondence between G-invariant connections on $F(M)$ and subspaces $\mathscr{H}_0 \subseteq T_{u_0}F(M)$ that satisfy the two conditions*

(1) $T_{u_0}F(M) = \mathscr{H}_0 \oplus \mathscr{V}_{u_0}$.
(2) $\mathscr{H}_0$ *is ψ-invariant.*

In other words, the representation ψ has $\mathscr{V}_{u_0}$ as an invariant subspace, and $\mathscr{H}_0$ is an invariant complement to $\mathscr{V}_{u_0}$.

Proof of Theorem. First suppose that $\mathscr{H}$ is an invariant connection. Let $u \in F(M)$ be an arbitrary point and let $p = \pi(u)$. Since G acts transitively on M, we can find $g \in G$ so that $gp_0 = p$. Then gu_0 and u are both in $F_p(M)$, so we can find $a \in \mathrm{GL}(V)$ so that $gu_0a = u$. See Figure 1.

But then $u = R_aL_gu_0$ and

$$\begin{aligned}R_aL_g\mathscr{H}_{u_0} &= R_a\mathscr{H}_{gu_0}\\ &= \mathscr{H}_{gu_0a}\\ &= \mathscr{H}_u.\end{aligned}$$

Thus, the whole connection $\mathscr{H}$ is determined by $\mathscr{H}_{u_0}$, it's value at u_0. We know from the discussion above that $\mathscr{H}_{u_0}$ satisfies properties (1) and and (2) of the theorem.

For the converse, suppose that we are give a subspace $\mathscr{H}_0$ with that has properties (1) and (2). We need to define the horizontal subspace $\mathscr{H}_u$ at every $u \in F(M)$.

Given u, let $p = \pi(u)$. We can write $u = gu_0a$ as before. We would then like to define

$$\mathscr{H}_u := R_aL_g\mathscr{H}_{u_0}, \tag{3.7}$$

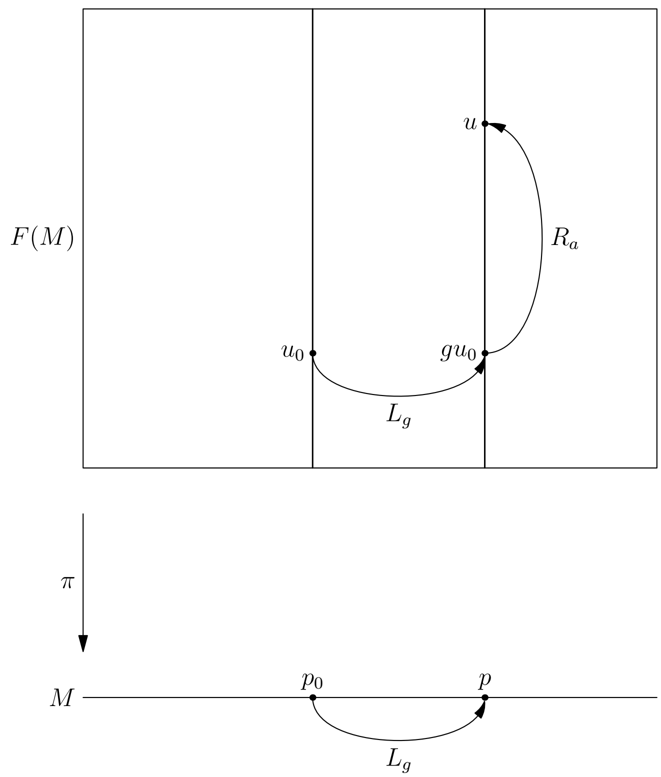


FIGURE 1. $u = gu_0a$

but we must check that this is independent of our choices. Suppose that we also have $u = g'u_0b$. Then $\pi(g'u_0b) = g'p_0 = p$. Thus $gp_0 = g'p_0$, so we must have $g' = gh$ for some $h \in H$. We then have

$$\begin{aligned} gu_0a &= u \\ &= g'u_0b \\ &= ghu_0b \\ &= gu_0\alpha(h)b. \end{aligned}$$

Since the action of $\mathrm{GL}(V)$ is free, this implies that $\alpha(h)b = a$, and so $b = \alpha(h^{-1})a$. But then we have

$$\begin{aligned} R_b L_{g'} \mathscr{H}_0 &= R_{\alpha(h^{-1})a} L_{gh} \mathscr{H}_0 \\ &= R_a R_{\alpha(h^{-1})} L_g L_h \mathscr{H}_0 \\ &= R_a L_g R_{\alpha(h^{-1})} L_h \mathscr{H}_0 \\ &= R_a L_g \{\psi(h) \mathscr{H}_0\} \\ &= R_a L_g \mathscr{H}_0, \end{aligned}$$

since $\mathscr{H}_0$ is ψ-invariant. Hence, we may define $\mathscr{H}_0$ consistently by (3.7). Since

$$R_a L_g \colon T_{u_0} F(M) \to T_u F(M)$$

is a linear isomorphism that preserves the vertical space, we have $T_u F(M) = \mathscr{H}_u \oplus \mathscr{V}_u$, from condition (1) of the Theorem. It's also easy to see that $\mathscr{H}_{u_0} = \mathscr{H}_0$.

It remains to check that $\mathscr{H}$ has the right invariance properties. To check that $\mathscr{H}$ is a connection, we must show

$$R_b \mathscr{H}_u = \mathscr{H}_{ub}, \qquad b \in \mathrm{GL}(V).$$

To see this, choose g and a so that $gu_0a = u$. Of course $ub = gu_0ab$. Then

$$\begin{aligned} R_b \mathscr{H}_u &= R_b \{R_a L_g \mathscr{H}_0\} \\ &= R_{ab} L_g \mathscr{H}_0 \\ &= \mathscr{H}_{gu_0ab} \\ &= \mathscr{H}_{ub}. \end{aligned}$$

To show that $\mathscr{H}$ is G-invariant, we must show

$$L_{g'} \mathscr{H}_u = \mathscr{H}_{g'u}, \qquad g' \in G.$$

Writing $u = gu_0a$, we have

$$\begin{aligned} L_{g'} \mathscr{H}_u &= L_{g'} \{R_a L_g \mathscr{H}_0\} \\ &= R_a L_{g'} L_g \mathscr{H}_0 \\ &= R_a L_{g'g} \mathscr{H}_0 \\ &= \mathscr{H}_{g'gu_0a} \\ &= \mathscr{H}_{g'u}. \end{aligned}$$

The proof is now complete.

□

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