EXAM

Exam #1

Math 3350
Summer II, 2000

July 21, 2000

ANSWERS

100 pts. **Problem 1**. In each part, find the general solution of the differential equation.

1.

$$\frac{dy}{dx} = x^2 e^{-y}$$

Answer:

We use the following sequence of steps.

$$\begin{gathered}\frac{dy}{dx} = x^2 e^{-y} \\ e^y \frac{dy}{dx} = x^2 \\ e^y\, dy = x^2\, dx \\ \int e^y\, dy = \int x^2\, dx \\ e^y = x^3/3 + C \\ \ln(e^y) = \ln(x^3/3 + C) \\ \boxed{y = \ln(x^3/3 + C)}\end{gathered}$$

2.

$$\frac{dy}{dx} = \frac{y^2 + xy + x^2}{x^2}, \qquad u = \frac{y}{x}$$

Answer:

Write the equation as

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1. \tag{1}$$

Since $u = y/x$, we have $y = xu$. Using the product rule, we get $y' = u + xu'$. Plugging into (1) we get

$$u + x\frac{du}{dx} = u^2 + u + 1$$

We solve this by the following sequence of steps.

$$\begin{aligned} u + x\frac{du}{dx} &= u^2 + u + 1 \\ x\frac{du}{dx} &= u^2 + 1 \\ x\,du &= (u^2+1)\,dx \\ \frac{du}{u^2+1} &= \frac{dx}{x} \\ \int \frac{du}{u^2+1} &= \int \frac{dx}{x} \\ \tan^{-1}(u) &= \ln|x| + C \\ u &= \tan(\ln|x| + C) \\ \frac{y}{x} &= \tan(\ln|x| + C) \\ \boxed{y = \tan(\ln|x| + C)}\,. \end{aligned}$$

3.

$$x\frac{dy}{dx} = x^2y^2 - y, \qquad v = xy$$

Answer:

Differentiating $v = xy$ with respect to x and using the product rule gives $v' = y + xy'$. Substitute $xy' = v' - y$ on the left-hand side of the equation and $x^2y^2 = v^2$ on the right. This gives the equation

$$\frac{dv}{dx} - y = v^2 - y.$$

Canceling the y's gives

$$\frac{dv}{dx} = v^2,$$

a differential equation for v that we solve by the following sequence of steps:

$$\begin{aligned}
\frac{dv}{dx} &= v^2 \\
dv &= v^2\,dx \\
\frac{dv}{v^2} &= dx \\
\int \frac{dv}{v^2} &= \int dx \\
-\frac{1}{v} &= x + C \\
\frac{1}{v} &= C - x \\
v &= \frac{1}{C - x}.
\end{aligned}$$

Since $v = xy$, this becomes

$$xy = \frac{1}{C - x},$$

and so

$$\boxed{y = \frac{1}{Cx - x^2}}.$$

4.

$$\frac{dy}{dx} + \frac{2}{x}y = x^4$$

Answer:
This is a first-order linear equation. Comparing with the standard form

$$\frac{dy}{dx} + p(x)y = r(x)$$

we see that $p(x) = 2/x$. Thus, the integrating factor is

$$\begin{aligned}
e^{\int p(x)\,dx} &= e^{\int (2/x)\,dx} \\
&= e^{2\ln|x|} \\
&= \left(e^{\ln|x|}\right)^2 \\
&= |x|^2 \\
&= x^2.
\end{aligned}$$

Multiplying both sides of the equation by x^2 gives

$$x^2 \frac{dy}{dx} + 2xy = x^6$$

The left-hand side is the same as

$$\frac{d}{dx}(x^2 y),$$

so the equation becomes

$$\frac{d}{dx}(x^2 y) = x^6.$$

Integrating both sides gives

$$x^2 y = \frac{1}{7}x^7 + C,$$

so dividing both sides by x^2 yields the solution

$$\boxed{y = \frac{1}{7}x^5 + \frac{C}{x^2}}.$$

5.

$$\frac{dy}{dx} + y = y^4, \qquad \text{(A Bernoulli equation)}$$

Answer:

Comparing with the standard form of a Bernoulli equation,

$$\frac{dy}{dx} + p(x)y = g(x)y^a,$$

we get $a = 4$. Thus, we should use the substitution

$$u = y^{1-a} = y^{1-4} = y^{-3}.$$

Differentiating this gives

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}.$$

Solving the original equation for dy/dx, we get

$$\begin{aligned} \frac{du}{dx} &= -3y^{-4}\frac{dy}{dx} \\ &= -3y^{-4}\left[y^4 - y\right] \\ &= -3 + 3y^{-3} \\ &= -3 + 3u \end{aligned}$$

Thus, we have

$$\frac{du}{dx} - 3u = -3,$$

a first-order linear equation for u. The integrating factor is e^{-3x}. Multiplying both sides by e^{-3x} we get

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}y = -3e^{-3x},$$

which is the same as

$$\frac{d}{dx}(e^{-3x}y) = -3e^{-3x}.$$

Integrating both sides gives

$$e^{-3x}u = e^{-3x} + C,$$

and so

$$u = 1 + Ce^{3x}.$$

Since $u = y^{-3}$, we have

$$y^{-3} = 1 + Ce^{3x},$$

and so, finally,

$$\boxed{y = \frac{1}{\sqrt[3]{1 + Ce^{3x}}}}.$$

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30 pts.

Problem 2. The following equation is exact. Solve it.

$$(2xy + y^2)\,dx + (x^2 + 2xy + y^2)\,dy = 0$$

Answer:
We look for a potential function $u(x, y)$, i.e., a function satisfying

$$\frac{\partial u}{\partial x} = 2xy + y^2 \quad (2)$$

$$\frac{\partial u}{\partial y} = x^2 + 2xy + y^2 \quad (3)$$

Integrating (2) with respect to x yields

$$u = x^2y + xy^2 + f(y), \quad (4)$$

where $f(y)$ is a function of y alone. Differentiating (4) gives

$$\frac{\partial u}{\partial y} = x^2 + 2xy + f'(y).$$

If this is going to come out the same as equation (3), we must have

$$x^2 + 2xy + f'(y) = x^2 + 2xy + y^2.$$

Thus, $f'(y) = y^2$, so $f(y) = y^3/3$. Plugging this back into equation (4), we have

$$u = x^2y + xy^2 + \frac{y^3}{3}.$$

The solutions of the original differential equation are the level curves of the potential function, so the solution to the differential equation is

$$\boxed{x^2y + xy^2 + \frac{y^3}{3} = C}$$

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30 pts.

Problem 3. The following equation is *not* exact. Find an integrating factor and solve the equation.

$$y\,dx + (x + xy)\,dy = 0$$

Answer:
Comparing to the standard form of the equation $P\,dx + Q\,dy = 0$, we have $P = y$ and $Q = x + xy$.

According to Theorem 2 of Section 1.5 of the book, an integrating factor F for the equation that is a function of y alone can be found from

$$\frac{1}{F}\frac{dF}{dy} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

provided the right-hand side is a function of y alone.

In our case,

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{y}\left(\frac{\partial}{\partial x}(x + xy) - \frac{\partial}{\partial y}(y)\right) = \frac{1}{y}(1 + y - 1) = 1,$$

which is a function of y alone. Thus, the equation for the integrating factor is

$$\frac{1}{F}\frac{dF}{dy} = 1,$$

and a solution is $F = e^y$. Thus, we can use e^y as an integrating factor.

Multiplying the equation by e^y yields

$$ye^y\,dx + (xe^y + xye^y)\,dy = 0$$

A potential function u must satisfy the equations

$$\frac{\partial u}{\partial x} = ye^y$$
$$\frac{\partial u}{\partial y} = xe^y + xye^y.$$

These equations can be solved by the method of the last problem, but in this case it's easy to see that $u = xye^y$ is a solution. Thus, the general solution of our differential equation is

$$\boxed{xye^y = C}$$

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30 pts. **Problem 4**. James Bond jumps skydives from a high-flying plane. Let the distance s he has traveled be measured downward from where he jumped, and let $v = ds/dt$.

James' mass is $m = 70$kg, so he is being pulled downwards by the force of gravity, $F = mg$.

He is also subject to the force of air resistance. Assume that the force of air resistance is proportional to the velocity v, say $F = kv$.

Find a differential equation for v. Be careful about the signs on the forces.

Solve this differential equation assuming that James' initial velocity is zero. Of course, the solution will still contain the unkown constant k.

Show that his velocity approaches a limiting value as $t \to \infty$, and find this limiting value (called the terminal velocity).

Answer:
The forces acting are the force of gravity mg (which is postive since we measure s downwards) and the force of air friction, which can be written as $-kv$ for some constant $k > 0$. We need the minus sign since this force acts in the opposite direction to v.

Using Newton's law $F = ma$, the equation of motion is

$$ma = mg - kv.$$

Using $a = dv/dt$, we can rewrite this equation as

$$m\frac{dv}{dt} + kv = mg$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = g \tag{5}$$

This can be solved either by separation of variables, or by considering it as a first order linear equation. If we take that latter approach, the integrating factor is $e^{\frac{k}{m}t}$. Multiplying by this integrating factor gives us

$$e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = ge^{\frac{k}{m}t}$$

which is the same as

$$\frac{d}{dt}(e^{\frac{k}{m}t}v) = ge^{\frac{k}{m}t}$$

Integrating both sides gives

$$e^{\frac{k}{m}t}v = \frac{mg}{k}e^{\frac{k}{m}t} + C.$$

From this we get the general solution of (5) as

$$v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}$$

Plugging in the condition that $v(0) = 0$ shows that $C = -mg/k$, so the solution is

$$\boxed{v(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t})}$$

Since $k/m > 0$, $e^{-\frac{k}{m}t} \to 0$ as $t \to \infty$. Thus the velocity approaches the limiting value mg/k as $t \to \infty$. This limiting value is called the terminal velocity.

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60 pts. **Problem 5**. In each part, find the general solution or solve the given initial value problem.

1.

$$y'' - 5y' + 6y = 0$$
$$y(0) = 0, \ y'(0) = 1$$

Answer:

The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$, which factors as $(\lambda - 2)(\lambda - 3) = 0$. Thus, the roots are $\lambda = 2, 3$. We are in the case of distinct real roots, so the general solution is

$$y = C_1e^{2x} + C_2e^{3x} \tag{6}$$

We must find the values of C_1 and C_2 so that the initial conditions are satisfied. Setting $t = 0$ in (6) and using the initial condition $y(0) = 0$ gives us the requirement

$$C_1 + C_2 = 0 \tag{7}$$

Differentiating (6) gives

$$y' = 2C_1e^{2x} + 3C_2e^{3x}$$

Setting $t = 0$ in this equation and using the initial condition $y'(0) = 1$ gives

$$2C_1 + 3C_2 = 1 \tag{8}$$

Thus, we must solve the system of equations given by (7) and (8). From (7), $C_2 = -C_1$. Plugging this into (8) gives $2C_1 - 3C_1 = 1$, so $C_1 = -1$ and $C_2 = 1$. Thus, the solution of our initial value problem is

$$\boxed{y = -e^{2x} + e^{3x}}$$

2.

$$y'' - 4y' + 4y = 0$$

Answer:
The characteristic equation is $0 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Thus, $\lambda = 2$ is a root of multiplicity two, and the general solution is

$$\boxed{y = C_1e^{2x} + C_2xe^{2x}}$$

3.

$$y'' - 4y' + 5y = 0$$

Answer:
The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$, which has roots

$$\begin{aligned}\lambda &= \frac{4 \pm \sqrt{(-4)^2 - 4(5)}}{2} \\ &= \frac{4 \pm \sqrt{16 - 20}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm 2i}{2} \\ &= 2 \pm i\end{aligned}$$

Thus, the general solution is

$$\boxed{y = C_1 e^{2x} \cos(x) + C_2 e^{2x} \sin(x)}$$
