
Problem Set

Assignment #1

Math 3350, Spring 2004

Feb. 6, 2004

ANSWERS

Problem 1. [Section 1.4, Problem 4] A rocket is shot straight up. During the initial stages of flight it has acceleration $7t \text{ m/s}^2$. The engine cuts out at $t = 10$ seconds. How high will the rocket go? (Neglect air resistance.)

Answer:

There are two phases of the flight, powered and unpowered. Let r be the distance of the rocket from the *center* of the earth. Then the velocity is $v = dr/dt$ and the acceleration is dv/dt .

During the powered phase of the flight, we are given that the acceleration is $7t \text{ m/s}^2$ (where $t = 0$ is the time the rocket is fired), so

$$\frac{dv}{dt} = 7t.$$

Integrating both sides of this equation gives

$$v = \frac{7}{2}t^2 + C.$$

Setting $t = 0$ in this equation shows that C is the velocity at $t = 0$. This is zero, since the rocket is not moving before we start the engine. Thus, we have

$$(1.1) \quad v = \frac{7}{2}t^2$$

and so

$$\frac{dr}{dt} = \frac{7}{2}t^2.$$

Integrating both sides gives

$$r = \frac{7}{6}t^3 + C.$$

Setting $t = 0$ in this equation shows that C is the value of r when $t = 0$. At $t = 0$, the rocket is on the surface of the earth, so $C = R$, where R is the radius of the earth. Thus, we have

$$(1.2) \quad r = \frac{7}{6}t^3 + R.$$

The engine cuts out at $t = 10$ s. Using (1.1), we get the velocity at engine cutoff to be

$$(1.3) \quad v_0 = \frac{7}{2}(10)^2 = 350 \text{ m/s}.$$

Using (1.2), we get the distance of the rocket from the center of the earth at engine cutoff to be

$$(1.4) \quad r_0 = \frac{7}{6}(10)^3 + R = \frac{3500}{3} + R.$$

Since $3500/3 \approx 1166.67 \text{ m} = 1.16667 \text{ km}$, the rocket is about one kilometer above the surface of the earth at engine cutoff. In example 4, the book gives the value of R as $R = 6372 \text{ km} = 6,372,000 \text{ m}$. Thus, we have

$$(1.5) \quad r_0 = \frac{3500}{3} + 6,372,000 = \frac{19119500}{3}$$

in meters.

In the unpowered phase of flight, only gravity acts on the rocket. So we need to figure out how far up an object that starts at $r = r_0$ with velocity $v = v_0$ will go. We use the techniques of example 4 in the book. Using the book's equation (9), we have

$$v \frac{dv}{dr} = -\frac{gR^2}{r^2}.$$

Separating the variables gives

$$v dv = -\frac{gR^2}{r^2} dr,$$

so integrating both sides gives

$$\frac{v^2}{2} = \frac{gR^2}{r} + C.$$

We may as well multiply both sides by 2 and rewrite this as

$$(1.6) \quad v^2 = \frac{2gR^2}{r} + C.$$

When $r = r_0$, we have $v = v_0$. Plugging this into (1.6) gives

$$v_0^2 = \frac{2gR^2}{r_0} + C,$$

so we have

$$C = v_0^2 - \frac{2gR^2}{r_0}.$$

Plugging this back into (1.6) gives

$$(1.7) \quad v^2 = \frac{2gR^2}{r} + v_0^2 - \frac{2gR^2}{r_0}.$$

The rocket will be at its highest point (greatest value of r) when it comes to a stop, just before starting to fall back to earth. Thus, we can set $v = 0$ in (1.7) and solve for r to find the greatest value of r reached by the rocket. Thus, we want to solve the equation

$$\frac{2gR^2}{r} + v_0^2 - \frac{2gR^2}{r_0} = 0$$

for r . A little algebra yields the solution

$$r = \frac{2gR^2r_0}{2gR - v_0^2r_0}.$$

So, to find the greatest value of r reached by the rocket, we plug the value of v_0 given by (1.3), the value of r_0 given by (1.5), $R = 6,372,000$ m and $g = 9.8$ m/s² into the equation above. The result is

$$r \approx 6.3794 \times 10^6 \text{ m} = 6,379.43 \text{ km}.$$

Thus, the rocket reaches about

$$6,379.43 - R = 7.43$$

kilometers above the surface of the earth.

It's worth remarking that if you ignore the fact that gravity gets weaker with distance and use a constant acceleration of $-g$ for the unpowered phase, you get an answer that is only a few meters off of the above.

Problem 2. [Section 1.4, Problem 14] A tank contains 400 gal of brine in which 100 lb of salt are dissolved. Fresh water runs into the tank at the rate of 2 gal/min, and the mixture, kept practically uniform by stirring, runs out at the same rate. How much salt will be left in the tank at the end of 1 hour?

Answer:

Let $y = y(t)$ be the number of pounds of salt in the tank at time t . We start the problem at $t = 0$ with 100 lb of salt, so $y(0) = 100$. As in Example 2 on page 20 of the book, dy/dt will be the rate that salt flows into the tank minus the rate at which salt flows out of the tank.

In this case, no salt flows into the tank. At time t the concentration of salt in the tank is $y/400$ pounds per gallon, and water is flowing out of the tank at 2 gallons per minute. Thus, the rate at which salt is flowing out is $(y/400)2$ pounds per minute. Thus, we have

$$\frac{dy}{dt} = 0 - \frac{y}{400}(2) = -\frac{1}{200}y.$$

This is an exponential decay equation, so the solution is

$$y = y_0e^{-t/200},$$

where $y_0 = y(0)$. From the information we're given, $y(0) = 100$, so the equation is

$$(2.1) \quad y = 100e^{-t/200}.$$

To find the amount of salt after 1 hour, we plug into this equation. Since we're measuring time in minutes in equation (2.1), we plug in $t = 60$. Thus, we have

$$y(60) = 100e^{-60/200} \approx 74.08 \text{ lb.}$$

Problem 3. [Section 1.5, Problem 32] Find an integrating factor and solve

$$(3.1) \quad 2xy \, dx + 3x^2 \, dy = 0.$$

Answer:

Comparing (3.1) with the standard form $P(x, y) \, dx + Q(x, y) \, dy = 0$, we have $P = 2xy$ and $Q = 3x^2$. We calculate that

$$\begin{aligned} \frac{\partial P}{\partial y} &= 2x \\ \frac{\partial Q}{\partial x} &= 6x. \end{aligned}$$

Since these are not the same, the equation is not exact.

Using Theorem 2 on page 30 of the book, we look for an integrating factor that is a function of y alone. We see that

$$\frac{1}{P} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \frac{1}{2xy} (6x - 2x) = \frac{2}{y}$$

is a function of y alone. Thus, our integrating factor is

$$\begin{aligned} F &= \exp\left(\int \frac{2}{y} \, dy\right) \\ &= \exp(2 \ln|y|) \\ &= y^2. \end{aligned}$$

Multiplying both sides of (3.1) by F gives us the equation

$$(3.2) \quad 2xy^3 \, dx + 3x^2y^2 \, dy = 0.$$

To solve this equation, we want to find a function $u = u(x, y)$ such that

$$(3.3) \quad \frac{\partial u}{\partial x} = 2xy^3$$

$$(3.4) \quad \frac{\partial u}{\partial y} = 3x^2y^2.$$

Integrating (3.3) with respect to x gives us $u = x^2y^3 + f(y)$ for some function $f(y)$. Taking the partial of this equation with respect to y gives

$$\frac{\partial u}{\partial y} = 3x^2y^2 + f'(y).$$

Comparing this with (3.4), we see that $f'(y) = 0$. Thus, $f(y)$ is a constant, which we can take to be zero (we just need one function u that satisfies (3.3) and (3.4)). Thus, we have $u = x^2y^3$. The solutions of (3.1) are the level curves of u , so the general solution of (3.1) is

$$(3.5) \quad x^2y^3 = C,$$

where C is an arbitrary constant.

It is also possible to solve this problem by finding an integrating factor that is a function of x alone, applying Theorem 1 in the book. Referring back to (3.1), we see that

$$\frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{1}{3x^2} (2x - 6x) = -\frac{4}{3} \frac{1}{x}$$

is a function of x alone. Thus our integrating factor is

$$\begin{aligned} F &= \exp\left(-\frac{4}{3} \int \frac{1}{x} dx\right) \\ &= \exp\left(-\frac{4}{3} \ln|x|\right) \\ &= \exp(\ln|x^{-4/3}|) \\ &= |x^{-4/3}| \\ &\sim x^{-4/3}, \end{aligned}$$

dropping the absolute value signs as usual (if this was not valid, we would discover that in the calculations below).

Multiplying (3.1) through by F , we get the equation

$$2x^{-1/3}y dx + 3x^{2/3} dy = 0.$$

Thus, we want a function u so that

$$(3.6) \quad \frac{\partial u}{\partial x} = 2x^{-1/3}y$$

$$(3.7) \quad \frac{\partial u}{\partial y} = 3x^{2/3}$$

Integrating (3.7) with respect to y gives us $u = 3x^{2/3}y + f(x)$ for some function $f(x)$. The partial of u with respect to x is

$$\frac{\partial u}{\partial x} = 2x^{-1/3}y + f'(x).$$

Comparing this with (3.6) we see $f'(x) = 0$, so $f(x)$ is a constant, which we can take to be zero. Thus, we have $u = 3x^{2/3}y$. This gives us the general solution of (3.1) as

$$(3.8) \quad 3x^{2/3}y = C.$$

This equation defines the same family of curves as (3.5). To see this, divide both sides of (3.8) by 3. The right-hand side is still an arbitrary constant, so (3.8) is the same as

$$x^{2/3}y = C.$$

We can cube both sides of this equation to get

$$x^2y^3 = C^3,$$

C^3 is an arbitrary constant, so this equation is equivalent to (3.5).

Problem 4. [Section 1.5, Problem 34] Find an integrating factor and solve

$$(4.1) \quad 2 \cos(y) dx = \sin(y) dy.$$

Answer:

Rewrite the equation in standard form as

$$(4.2) \quad 2 \cos(y) dx - \sin(y) dy = 0,$$

so $P = 2 \cos(y)$ and $Q = -\sin(y)$. We have

$$\begin{aligned} \frac{\partial P}{\partial y} &= -2 \sin(y) \\ \frac{\partial Q}{\partial x} &= 0, \end{aligned}$$

so the equation is not exact. We calculate that

$$\frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = -\frac{1}{\sin(y)} (-2 \sin(y)) = 2,$$

which is a function of x alone. Thus, by Theorem 1 on page 30 of the book, we have an integrating factor that is a function of x alone, which is given by

$$F = \exp\left(\int 2 dx\right) = e^{2x}.$$

Multiplying (4.2) through by e^{2x} gives the equation

$$2e^{2x} \cos(y) dx - e^{2x} \sin(y) dy = 0.$$

It's easy to see that the left-hand side of this equation is du for $u = e^{2x} \cos(y)$, so the general solution of (4.1) is

$$e^{2x} \cos(y) = C,$$

where C is an arbitrary constant.
