
EXAM

Exam #3

Math 2360, Spring 2001

April 24, 2001

ANSWERS

40 pts.

Problem 1. In this problem, we will work in the vectorspace

$$P_3 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \},$$

the space of polynomials of degree less than 3. Let \mathcal{P} be the ordered basis $\mathcal{P} = [x^2 \ x \ 1]$ of P_3 and let \mathcal{Q} be the ordered basis

$$\mathcal{Q} = [2x^2 + x + 1 \ x^2 + 2x + 1, x^2 + x + 1]$$

of P_3 .

A. Find the change of basis matrices $S_{\mathcal{P}\mathcal{Q}}$ and $S_{\mathcal{Q}\mathcal{P}}$.

Answer:

The defining equation of the change of basis matrix $S_{\mathcal{P}\mathcal{Q}}$ is $\mathcal{Q} = \mathcal{P}S_{\mathcal{P}\mathcal{Q}}$. Reading off the coefficients, we have

$$\underbrace{[2x^2 + x + 1 \ x^2 + 2x + 1, x^2 + x + 1]}_{\mathcal{Q}} = \underbrace{[x^2 \ x \ 1]}_{\mathcal{P}} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

(a matrix multiplication), so

$$S_{\mathcal{P}\mathcal{Q}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (1)$$

To find $S_{\mathcal{Q}\mathcal{P}}$, we use the fact that $S_{\mathcal{Q}\mathcal{P}} = S_{\mathcal{P}\mathcal{Q}}^{-1}$. Thus,

$$S_{\mathcal{Q}\mathcal{P}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

where we have used a calculator to find the inverse of the matrix in (1).

B. Let $p(x) = 2x^2 - x + 1$. Find $[p(x)]_{\mathcal{Q}}$, the coordinate vector of $p(x)$ with respect to the ordered basis \mathcal{Q} . Show how $p(x)$ can be written as a linear combination to the elements of \mathcal{Q}

Answer:

First we find $[p(x)]_{\mathcal{P}}$, since that's easy. The defining equation of $[p(x)]_{\mathcal{P}}$ is $p(x) = \mathcal{P}[p(x)]_{\mathcal{P}}$. We have

$$p(x) = 2x^2 - x + 1 = \underbrace{[x^2 \ x \ 1]}_{\mathcal{P}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

and so

$$[p(x)]_{\mathcal{P}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

To find $[p(x)]_{\mathcal{Q}}$, we use the change of coordinate equation $[p(x)]_{\mathcal{Q}} = S_{\mathcal{Q}\mathcal{P}}[p(x)]_{\mathcal{P}}$, so

$$[p(x)]_{\mathcal{Q}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix},$$

using our value of $S_{\mathcal{Q}\mathcal{P}}$ from above. Thus,

$$\boxed{[p(x)]_{\mathcal{Q}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}.$$

The defining equation of $[p(x)]_{\mathcal{Q}}$ is $p(x) = \mathcal{Q}[p(x)]_{\mathcal{Q}}$. Thus, we have

$$\begin{aligned} p(x) &= \mathcal{Q}[p(x)]_{\mathcal{Q}} \\ &= \underbrace{[2x^2 + x + 1 \quad x^2 + 2x + 1, x^2 + x + 1]}_{\mathcal{Q}} \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{[p(x)]_{\mathcal{Q}}} \\ &= 1(2x^2 + x + 1) - 2(x^2 + 2x + 1) + 2(x^2 + x + 1). \end{aligned}$$

Thus, we can write $p(x)$ as a linear combination of the elements of \mathcal{Q} as follows

$$\boxed{p(x) = 1(2x^2 + x + 1) - 2(x^2 + 2x + 1) + 2(x^2 + x + 1)}.$$

40 pts.

Problem 2. In this problem, we will work in the vectorspace

$$P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\},$$

the space of polynomials of degree less than 3. Let \mathcal{P} be the ordered basis $\mathcal{P} = [x^2 \quad x \quad 1]$ of P_3 .

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & 8 \end{bmatrix},$$

which is invertible. Let \mathcal{Q} be the ordered basis defined by $\mathcal{Q} = \mathcal{P}A$. Let $T: P_3 \rightarrow P_3$ be the linear transformation defined by $T(p(x)) = p'(x) + p(x)$.

A. Find $[T]_{\mathcal{P}\mathcal{P}}$, the matrix of T with respect to the ordered basis \mathcal{P} .

Answer:

The defining equation of $[T]_{\mathcal{P}\mathcal{P}}$ is

$$T(\mathcal{P}) = \mathcal{P}[T]_{\mathcal{P}\mathcal{P}}. \quad (2)$$

and $T(\mathcal{P}) = [T(x^2) \quad T(x) \quad T(1)]$. Since $T(p(x)) = p'(x) + p(x)$ we have

$$T(x^2) = (x^2)' + x^2 = 2x + x^2$$

$$T(x) = (x)' + x = 1 + x$$

$$T(1) = (1)' + 1 = 0 + 1 = 1,$$

so $T(\mathcal{P}) = [x^2 + 2x \quad x + 1 \quad 1]$. Reading off the coefficients, we have

$$\underbrace{[x^2 + 2x \quad x + 1 \quad 1]}_{T(\mathcal{P})} = \underbrace{[x^2 \quad x \quad 1]}_{\mathcal{P}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

(matrix multiplication). Comparing this with equation (2), we see that

$$\boxed{[T]_{\mathcal{P}\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}.$$

B. Find $[T]_{\mathcal{Q}\mathcal{Q}}$, the matrix of T with respect to the basis \mathcal{Q} .

Answer:

We will use the change of basis equation for linear transformations, i.e.,

$$[T]_{\mathcal{Q}\mathcal{Q}} = S_{\mathcal{Q}\mathcal{P}}[T]_{\mathcal{P}\mathcal{P}}S_{\mathcal{P}\mathcal{Q}}.$$

Since $S_{\mathcal{Q}\mathcal{P}} = S_{\mathcal{P}\mathcal{Q}}^{-1}$, we can rewrite this equation as

$$[T]_{\mathcal{Q}\mathcal{Q}} = S_{\mathcal{P}\mathcal{Q}}^{-1}[T]_{\mathcal{P}\mathcal{P}}S_{\mathcal{P}\mathcal{Q}}. \quad (3)$$

The defining equation of $S_{\mathcal{P}\mathcal{Q}}$ is

$$\mathcal{Q} = \mathcal{P}S_{\mathcal{P}\mathcal{Q}}. \quad (4)$$

Since the basis \mathcal{Q} is defined by $\mathcal{Q} = \mathcal{P}A$, where A is given in the problem, comparison with equation (4) shows that $S_{\mathcal{P}\mathcal{Q}} = A$. Hence, plugging into (3) shows that

$$[T]_{\mathcal{Q}\mathcal{Q}} = A^{-1}[T]_{\mathcal{P}\mathcal{P}}A.$$

Using the given value of A and the value of $[T]_{\mathcal{P}\mathcal{P}}$ from the previous part of the problem, we have

$$[T]_{\mathcal{Q}\mathcal{Q}} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & 8 \end{bmatrix}.$$

Using a calculator for the matrix computations, we find

$$[T]_{\mathcal{Q}\mathcal{Q}} = \begin{bmatrix} -1 & -3 & 7 \\ 2 & 5/2 & 11/2 \\ 0 & 1/2 & 3/2 \end{bmatrix}.$$

60 pts.

Problem 3. Let $\mathcal{U} = [\mathbf{u}_1 \ \mathbf{u}_2]$ be the ordered basis of \mathbb{R}^2 where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis \mathcal{E} is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

in other words, $T(\mathbf{x}) = A\mathbf{x}$.

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

Answer:

The defining equation of $S_{\mathcal{E}\mathcal{U}}$ is $\mathcal{U} = \mathcal{E}S_{\mathcal{E}\mathcal{U}}$. The corresponding matrix equation is

$$\text{mat}(\mathcal{U}) = \text{mat}(\mathcal{E})S_{\mathcal{E}\mathcal{U}} = IS_{\mathcal{E}\mathcal{U}} = S_{\mathcal{E}\mathcal{U}}$$

Thus, we have

$$S_{\mathcal{E}\mathcal{U}} = \text{mat}(\mathcal{U}) = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

Thus,

$$S_{\mathcal{E}\mathcal{U}} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

We know that $S_{\mathcal{U}\mathcal{E}} = S_{\mathcal{E}\mathcal{U}}^{-1}$. Using a calculator for the computation of the inverse, we have

$$S_{\mathcal{U}\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ -1/2 & 1 \end{bmatrix}.$$

B. Find $[T]_{\mathcal{U}\mathcal{U}}$, the matrix of T with respect to \mathcal{U} .

Answer:

We are given the matrix of T with respect to the standard basis, so

$$[T]_{\mathcal{E}\mathcal{E}} = A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

To find $[T]_{\mathcal{U}\mathcal{U}}$, we use the change of coordinate equation for linear transformations, i.e.,

$$\begin{aligned} [T]_{\mathcal{U}\mathcal{U}} &= S_{\mathcal{U}\mathcal{E}}[T]_{\mathcal{E}\mathcal{E}}S_{\mathcal{E}\mathcal{U}} \\ &= \begin{bmatrix} 1 & -1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus, the answer to this part of the problem is

$$\boxed{[T]_{\mathcal{U}\mathcal{U}} = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}}.$$

C. Find the scalars c_1 and c_2 such that

$$T(2\mathbf{u}_1 - 3\mathbf{u}_2) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

Answer:

Let $\mathbf{v} = 2\mathbf{u}_1 - 3\mathbf{u}_2$. Then, of course,

$$\mathbf{v} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}}_{\mathcal{U}} \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

The defining equation of $[\mathbf{v}]_{\mathcal{U}}$ is $\mathbf{v} = \mathcal{U}[\mathbf{v}]_{\mathcal{U}}$, so we have

$$[\mathbf{v}]_{\mathcal{U}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

The equation for the action of the linear transformation T in the \mathcal{U} -coordinates is

$$[T(\mathbf{v})]_{\mathcal{U}} = [T]_{\mathcal{U}\mathcal{U}}[\mathbf{v}]_{\mathcal{U}}.$$

Thus, we have

$$[T(\mathbf{v})]_{\mathcal{U}} = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

The defining equation of $[T(\mathbf{v})]_{\mathcal{U}}$ is

$$T(\mathbf{v}) = \mathcal{U}[T(\mathbf{v})]_{\mathcal{U}}$$

So we have

$$T(2\mathbf{u}_1 - 3\mathbf{u}_2) = T(\mathbf{v}) = \underbrace{[\mathbf{u}_1 \quad \mathbf{u}_2]}_{\mathcal{U}} \underbrace{\begin{bmatrix} -8 \\ 3 \end{bmatrix}}_{[T(\mathbf{v})]_{\mathcal{U}}} = -8\mathbf{u}_1 + 3\mathbf{u}_2,$$

So the answer is

$$\boxed{c_1 = -8, \quad c_2 = 3}.$$

40 pts.

Problem 4. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.$$

Find the characteristic polynomial of A and the eigenvalues of A .

Answer:

We have

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 3 \\ 2 & -\lambda \end{bmatrix}.$$

Thus, the characteristic polynomial $p(\lambda)$ of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1 - \lambda & 3 \\ 2 & -\lambda \end{vmatrix} \\ &= (1 - \lambda)(-\lambda) - 6 \\ &= -\lambda + \lambda^2 - 6 \\ &= \lambda^2 - \lambda - 6 \end{aligned}$$

Thus, we have

$$\boxed{p(\lambda) = \lambda^2 - \lambda - 6}.$$

The eigenvalues of A are the roots of the characteristic polynomial. The characteristic polynomial factors as $p(\lambda) = (\lambda - 3)(\lambda + 2)$, so the roots are 3 and -2 . Thus,

$$\boxed{\text{Eigenvalues of } A = -2, 3}.$$

60 pts.

Problem 5. In each part you are given an matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A . Determine if A is diagonalizable, and if it is, find a matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

A.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \text{Eigenvalues} = 1, 2.$$

Answer:

Consider first the eigenvalue $\lambda = 1$. We have

$$A - \lambda I = A - I = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

The RREF of this matrix is (by calculator)

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace E_1 is the nullspace of $A - I$, which is the same as the nullspace of R , i.e., the solution space of $R\mathbf{x} = \mathbf{0}$. Calling the variables x_1 , x_2 and x_3 , we see that x_3 is a free variable, say $x_3 = \alpha$. The second row of R gives the equation $x_2 - x_3 = 0$, so $x_2 = \alpha$ and the first row gives the equation $x_1 - x_3 = 0$, so $x_1 = \alpha$. Thus, the vectors in the nullspace of R are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, a basis of the eigenspace E_1 is

$$\boxed{\text{Basis of } E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}.$$

so E_1 has dimension one.

Next, consider the eigenvalue $\lambda = 2$. We have

$$A - \lambda I = A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix},$$

which has the RREF

$$R = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace E_2 is the nullspace of $A - 2I$, which is the same as the nullspace of R . Abbreviating the computation of the nullspace of R , we have

$$\begin{aligned} x_3 &= \alpha \\ x_2 - \frac{3}{2}x_3 &= 0 \implies x_2 = \frac{3}{2}\alpha \\ x_1 - \frac{1}{2}x_3 &= 0 \implies x_1 = \frac{1}{2}\alpha \end{aligned}$$

so the nullspace is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\alpha \\ \frac{1}{2}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Thus, we have

$$\boxed{\text{Basis of } E_2 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}}.$$

Alternatively, we could multiply this vector by 2 and say

$$\boxed{\text{Basis of } E_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$$

and so avoid the agony of dealing with fractions.

We have only produced 2 independent eigenvectors, and there are no additional eigenvectors that are independent of these two. Thus, A does not have a basis of eigenvectors, we conclude

$$\boxed{A \text{ is not diagonalizable}}.$$

B.

$$A = \begin{bmatrix} 10 & -3 & -6 \\ 12 & -5 & -6 \\ 12 & -3 & -8 \end{bmatrix}, \quad \text{Eigenvalues} = -2, 1.$$

Answer:

First consider the eigenvalue $\lambda = -2$. Then we have

$$A - \lambda I = A + 2I = \begin{bmatrix} 12 & -3 & -6 \\ 12 & -3 & -6 \\ 12 & -3 & -6 \end{bmatrix},$$

which has the RREF

$$R = \begin{bmatrix} 1 & -1/4 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the abbreviated computation of the nullspace we have

$$\begin{aligned} x_2 &= \alpha \\ x_3 &= \beta \\ x_1 - \frac{1}{4}x_2 - \frac{1}{2}x_3 &= 0 \implies x_1 = \frac{1}{4}\alpha + \frac{1}{2}\beta. \end{aligned}$$

The parametrization of the nullspace is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, we get

$$\boxed{\text{Basis of } E_{-2} = \begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}}. \quad (5)$$

As matter of personal taste, I'll get rid of the fractions by multiplying the first vector by 4 and the second by 2. That gives me

$$\boxed{\text{Basis of } E_{-2} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}. \quad (6)$$

We see that E_{-2} has dimension 2.

Next, consider the eigenvalue $\lambda = 1$. We have

$$A - \lambda I = A - I = \begin{bmatrix} 9 & -3 & -6 \\ 12 & -3 & -9 \\ 12 & -3 & -9 \end{bmatrix},$$

which has the RREF

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that we have already computed the nullspace of this matrix R in the first part of the problem. Thus, we know

$$\boxed{\text{Basis of } E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}. \quad (7)$$

Since we've found three linearly independent eigenvectors, A has a basis of eigenvectors and

A is diagonalizable.

Using the basis (6) of E_{-2} as a matter of taste (I could equally well use (5)), and the basis (7) of E_1 , set

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then P is invertible, D is a diagonal matrix and $P^{-1}AP = D$ (check it if you don't believe me).

40 pts.

Problem 6. In each part, determine if the given transformation T is linear. If it is linear, find a matrix so that $T(\mathbf{x}) = A\mathbf{x}$. If T is not linear, justify your answer.

A. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

Answer:

For T to be linear, it must preserve scalar multiplication, i.e., it must be true that $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$ for all scalars α and all $\mathbf{x} \in \mathbb{R}^2$. We have

$$T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} (\alpha x_1)(\alpha x_2) \\ (\alpha x_1)^2 + (\alpha x_2)^2 \end{bmatrix} = \begin{bmatrix} \alpha^2 x_1 x_2 \\ \alpha^2 (x_1^2 + x_2^2) \end{bmatrix} \quad (8)$$

$$\alpha T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \alpha \begin{bmatrix} x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 x_2 \\ \alpha (x_1^2 + x_2^2) \end{bmatrix}. \quad (9)$$

If T was linear, the right hand sides of equations (8) and (9) would have to be equal for all values of α , x_1 and x_2 . In particular, we would have to have

$$\alpha^2 x_1 x_2 = \alpha x_1 x_2$$

for all values of α , x_1 and x_2 . This is plainly not the case (e.g., $\alpha = 2$, $x_1 = x_2 = 1$), so

T is **not** linear.

B. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 5x_3 \\ x_1 - x_3 \end{bmatrix}$$

Answer:

We can rewrite the right hand side of the definition of T as a matrix multiplication

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where A is the matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -1 \end{bmatrix}.$$

Since T is given by matrix multiplication, T must be linear.

$$\boxed{T \text{ is linear}}.$$

50 pts.

Problem 7. Let S be the subspace of \mathbb{R}^4 spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

- A. Cut down the list of vectors above to a basis for S . What is the dimension of S ?
- B. For each of the following vectors, determine if the vector is in S and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} -3 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

Answer:

We can do all the computation in one RREF calculation on the calculator. Make up a matrix containing all these vectors

$$A = \left[\begin{array}{cccc|cc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{w}_1 & \mathbf{w}_2 \\ 2 & -1 & 8 & 1 & -3 & 5 \\ 2 & 3 & 0 & 1 & 5 & 3 \\ 1 & -1 & 5 & 1 & -2 & 4 \\ 1 & 3 & -3 & 0 & -1 & 0 \end{array} \right]$$

The RREF of this is

$$R = \left[\begin{array}{cccc|cc} 1 & 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

Columns 1, 2 and 4 of R form a basis for the span of the columns left of the bar, so the same is true of A . Thus,

$$\boxed{\mathbf{v}_1, \mathbf{v}_2 \text{ and } \mathbf{v}_4 \text{ are a basis of } S}.$$

Since S has a basis with three elements,

$$\boxed{\text{The dimension of } S \text{ is } 3}.$$

In R , we have $\text{col}_5(R) = -\text{col}_1(R) + 2\text{col}_2(R) + \text{col}_4(R)$. The same relation must hold among the columns of A , so we have

$$\boxed{\mathbf{w}_1 = -\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4}.$$

Since \mathbf{w}_1 is a linear combination of vectors in S ,

$$\boxed{\mathbf{w}_1 \in S}$$

The last column of R is not a linear combination of the columns to the left of the bar (because of the last row), so the same is true of A . Thus,

$$\boxed{\mathbf{w}_2 \notin S}.$$