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EXAM

Exam #1

Math 2360, Second Summer Session, 2002

April 24, 2001

ANSWERS

50 pts. **Problem 1**. In each part you are given the augmented matrix of a system of linear equations, with the coefficent matrix in reduced row echelon form. Determine if the system is consistent and, if it is consistent, find all solutions.

A.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Answer: $(x, y, z) = (1, 2, -5)$

B.

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & 1 \\ 0 & 1 & -3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Answer:

Call the variables $x_1, \ldots, x_5$. Columns 1,2 and 4 of the coefficent matrix contain leading entries, so x_1, x_2 and x_4 are nonleading variables and x_3 and x_5 are free variables. I'll use α and β for the free parameters, say $x_3 = \alpha$, $x_5 = \beta$. Reading the matrix from the bottom up, we get the equations

$$\begin{aligned} x_4 + 5x_5 = 4 &\implies x_4 = 4 - 5\beta \\ x_2 - 3x_3 + 2x_5 = 2 &\implies x_2 = 2 + 3\alpha - 2\beta \\ x_1 - 2x_3 - x_5 = 1 &\implies x_1 = 1 + 2\alpha + \beta \end{aligned}$$

Hence, the solution set of the system is parametrized by

$$(x_1, x_2, x_3, x_4, x_5) = (1 + 2\alpha + \beta, 2 + 3\alpha - 2\beta, \alpha, 4 - 5\beta, \beta).$$

C.

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & 4 \\ 0 & 1 & -3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{array}\right].$$

Answer: The system is inconsistent.

40 pts.

Problem 2. In each part, solve the linear system using the Gauss-Jordan method (i.e., reduce the coefficent matrix to Reduced Row Echelon Form). Show the augmented matrix you start with and the augmented matrix you finish with. It's not necessary to show individual row operations, you can just find the Reduced Row Echelon Form with your calculator.

A.

$$\begin{array}{rcrcrcr} 4x & + & 2y & + & z & = & 9 \\ 2x & + & 2y & & & = & 2 \\ 2x & + & 2y & + & z & = & 5 \end{array}$$

Answer:

The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 9 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 5 \end{array}\right].$$

After reducing the coefficent matrix to RREF, we get the following matrix

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The linear system corresponding to this matrix is

$$x = 2$$
$$y = -1$$
$$z = 3$$

so the solution is $(x, y, z) = (2, -1, 3)$.

B.

$$\begin{array}{rcrcrcr} 2x & + & y & - & z & = & 4 \\ x & + & y & + & z & = & 3 \\ 3x & + & 2y & & & = & 7 \end{array}$$

Answer:

The augmented matrix is

$$\left[\begin{array}{ccr|c} 2 & 1 & -1 & 4 \\ 1 & 1 & 1 & 3 \\ 3 & 2 & 0 & 7 \end{array}\right].$$

Putting the coefficent matrix in RREF gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Form the last row, the system is consistent. We see that x and y are leading variables and that z is a free variable, say $z = \alpha$. The second row of the matrix gives the equation $y + 3z = 2$, so $y = 2 - 3\alpha$. The first row gives the equation $x - 2z = 1$ and so $x = 1 + 2\alpha$. We conclude that the solution set of this system is parametrized by

$$(x, y, z) = (1 + 2\alpha, 2 - 3\alpha, \alpha).$$

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50 pts.

Problem 3.
Consider row operations on matrices with 4 rows.

A. Consider the row operation $R_2 \leftrightarrow R_4$.

i. Find the elementary matrix E corresponding to this row operation.

Answer: $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

ii. Find the inverse row operation.

Answer: $R_2 \leftrightarrow R_4$.

iii. From the last part, find E^{-1}.

Answer: $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

B. Consider the row operation $R_3 \leftarrow 7R_3$.

i. Find the elementary matrix E corresponding to this row operation.

Answer: $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ii. Find the inverse row operation.

Answer: $R_3 \leftarrow \frac{1}{7}R_3$

iii. From the last part, find E^{-1}.

Answer: $E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. Consider the row operation $R_1 \leftarrow R_1 + 5R_3$.

i. Find the elementary matrix E corresponding to this row operation.

Answer: $E = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ii. Find the inverse row operation.

Answer: $R_1 \leftarrow R_1 - 5R_3$.

iii. From the last part, find E^{-1}.

Answer: $E^{-1} = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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40 pts.

Problem 4. In each part, use row operations to determine if the matrix A is invertible and, if so, to find the inverse. It is not necessary to show the individual row operations (you can just use the `rref` key on the calculator). Show the augmented matrix you start with and the augmented matrix you finish with. Give the matrix entries in fractional form.

A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Answer:

To start, we form the augmented matrix

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right].$$

We then perform row operations to get the part of this matrix to the left of the bar in reduced row echelon form. The result is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & -1 & -2 \end{array}\right].$$

Since the RREF of A is the identity, we conclude that A is invertible. The inverse of A is the part of the last matrix to the right of the bar, so

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1/2 \\ 2 & -1 & -2 \end{bmatrix}$$

B.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Answer:

We begin by forming the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

We then perform row operations to put the left part of this matrix in reduced row echelon form. The result is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{array}\right].$$

Since the RREF of A is not the identity, we conclude that A **is not invertible**.

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40 pts.

Problem 5. The matrix

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 6 \end{bmatrix}$$

is invertible. Express A as a product of elementary matrices. (You can use your calculator for the row operations.)

Answer:
In each line of the following table we show the matrix we have arrived at, the row operation to be preformed next, and the elementary matrix for this row operation.

$$
\begin{array}{rll}
A = \begin{bmatrix} 0 & 3 \\ 2 & 6 \end{bmatrix} & R_1 \leftrightarrow R_2 & E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
E_1 A = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix} & R_1 \leftarrow R_1 - 2R_2 & E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\
E_2 E_1 A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} & R_1 \leftarrow \frac{1}{2} R_1 & E_3 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \\
E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} & R_2 \leftarrow \frac{1}{3} R_2 & E_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \\
E_4 E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & &
\end{array}
$$

From the last line, we have

$$(*) \qquad E_4 E_3 E_2 E_1 A = I$$

The inverses of the elementary matrices above are

$$
\begin{aligned}
E_1^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
E_2^{-1} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
E_3^{-1} &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
E_4^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.
\end{aligned}
$$

From $(*)$, we have

$$
\begin{aligned}
A &= (E_4 E_3 E_2 E_1)^{-1} \\
&= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.
\end{aligned}
$$

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40 pts.

Problem 6. Find the following determinant by the method of elimination, i.e., by using row operations and keeping track of the effect of the row operations on the determinant. Show the row operations you use and the intermediate determinants (you use a calculator to perform the row ops).

Sorry, no credit for finding it by another method.

$$\begin{vmatrix} 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 4 & 0 \\ 5 & 5 & 5 & 0 \end{vmatrix}$$

Answer:
In each line below, we show the next step in the equation and the row operation to be performed next.

$$\begin{vmatrix} 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 4 & 0 \\ 5 & 5 & 5 & 0 \end{vmatrix} \qquad R_1 \leftrightarrow R_3$$

$$= -\begin{vmatrix} 2 & 2 & 4 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 5 & 5 & 5 & 0 \end{vmatrix} \qquad R_1 \leftarrow \frac{1}{2}R_1$$

$$= -2\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 5 & 5 & 5 & 0 \end{vmatrix} \qquad R_4 \leftarrow R_4 - 5R_1$$

$$= -2\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -5 & 0 \end{vmatrix} \qquad R_2 \leftrightarrow R_3$$

$$= 2\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{vmatrix} \qquad R_2 \leftarrow \frac{1}{2}R_2$$

$$= 4\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{vmatrix} \qquad R_3 \leftarrow R_3 - 2R_2$$

$$= 4\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -5 & 0 \end{vmatrix} \qquad R_4 \leftarrow R_4 - 5R_3$$

$$= 4\begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= 4(1)(1)(-1)(5)$$
$$= \boxed{-20}.$$

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70 pts.

Problem 7. Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

A. Find the cofactors A_{12} and A_{33}.

Answer:

$$A_{12} = (-1)^{(1+2)} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -[1(1) - (2)(2)] = 3$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = (0)(3) - (1)(3) = -3$$

B. Compute $\det(A)$, using the cofactor expansion along a selected row or column.

Answer:

I'll expand along the top row. This gives

$$\begin{aligned}\det(A) &= 0A_{11} + 3A_{12} + 1A_{13} \\ &= 3\left\{(-1)^{(1+2)}\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}\right\} + 1\left\{(-1)^{1+3}\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}\right\} \\ &= 3\left\{-[(1)(1)-(2)(2)]\right\} + (1)(1) - (2)(3) \\ &= 3(3) - 5 \\ &= \boxed{4}\end{aligned}$$

C. Find the adjoint matrix adj(A) from it's definition in terms of cofactors.

Answer:

From the definition in terms of cofactors, we have

$$\begin{aligned}\operatorname{adj} A &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}\end{bmatrix}^T \\ &= \begin{bmatrix} +\begin{vmatrix}3 & 2\\ 1 & 1\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} & +\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix} \\ -\begin{vmatrix}3 & 1\\ 1 & 1\end{vmatrix} & +\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} & -\begin{vmatrix}0 & 3\\ 2 & 1\end{vmatrix} \\ +\begin{vmatrix}3 & 1\\ 3 & 2\end{vmatrix} & -\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix} & +\begin{vmatrix}0 & 3\\ 1 & 3\end{vmatrix}\end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 3 & -5 \\ -2 & -2 & 6 \\ 3 & 1 & -3\end{bmatrix}^T \\ &= \begin{bmatrix} 1 & -2 & 3 \\ 3 & -2 & 1 \\ -5 & 6 & -3\end{bmatrix}\end{aligned}$$

D. **Use the information above** to find A^{-1}.

Answer:

We know that

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)}\operatorname{adj}(A) \\ &= \frac{1}{4}\begin{bmatrix} 1 & -2 & 3 \\ 3 & -2 & 1 \\ -5 & 6 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1/4 & -1/2 & 3/4 \\ 3/4 & -1/2 & 1/4 \\ -5/4 & 3/2 & -3/4 \end{bmatrix} \end{aligned}$$

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40 pts.

Problem 8. Let A be an $n \times n$ matrix. If $A^2 = I$, what are the possible values of $\det(A)$? Justify your answer.

Answer:
We know that $\det(AB) = \det(A)\det(B)$ for square matrices A and B. Thus, $\det(A^2) = \det(AA) = \det(A)\det(A) = \det(A)^2$.

Now suppose that $A^2 = I$. Taking the determinant of both sides yields $\det(A^2) = 1$. By the last paragraph, this gives us $\det(A)^2 = 1$.

Thus $\det(A)$ is a solution of the equation $x^2 = 1$. The solutions of this equation are $+1$ and -1. So we conclude that the value of $\det(A)$ must be 1 or -1.

By the way, $A^2 = I$ does not imply that $A = I$. For example, any elementary matrix E of Type I satisfies $E^2 = I$, $E \neq I$, $\det(E) = -1$.

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40 pts.

Problem 9. **Use Cramer's rule** to solve the following system.

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 1 \\ & & y & - & 2z & = & 0 \\ 2x & & + & & z & = & 0 \end{array}$$

Answer:
In matrix form, the system is $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Expanding the determinant of A along the first column gives

$$\det(A) = 2\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -4.$$

By Cramer's rule, we have

$$x = \frac{\det(A_1)}{\det(A)} = -\frac{1}{4}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = -\frac{1}{4}$$

$$y = \frac{\det(A_2)}{\det(A)} = -\frac{1}{4}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 2 & 0 & 1 \end{vmatrix} = -\frac{1}{4}\left\{-\begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix}\right\} = 1$$

$$z = \frac{\det(A_3)}{\det(A)} = -\frac{1}{4}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = \frac{1}{2}.$$

Here A_j is the matrix obtained by replacing column j of A with the column vector $\mathbf{b}$. In each case I have expanded the determinant of A_j along column j.

From this calculation, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/2 \end{bmatrix}$$

as the solution to the system.
