EXAM

Exam #1

Math 2360
Summer II, 2000
Morning Class

Oct. 11, 2000

ANSWERS

50 pts.

Problem 1. In each part, you are given the augmented matrix of a linear system. The coefficient matrix is already in Reduced Row Echelon Form. Determine if the system is consistent, and if it is consistent find all solutions.

(a.)

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Answer: $(x, y, z) = (5, -1, 3)$

(b.)

$$\left[\begin{array}{crcrc|c} 1 & -1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Answer: The system is **inconsistent**, because of the last row.

(c.)

$$\left[\begin{array}{crcrc|r} 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Answer:

The system is consistent. Call the variables x_1 through x_5. The variables x_1,x_3 and x_5 are leading variables and x_2 and x_4 are non-leading or free variables. Say $x_2 = \alpha$ and $x_4 = \beta$. The third row of the matrix gives us the equation $x_5 = -3$. The second row of the matrix gives the equation $x_3 - 2x_4 = 0$, so $x_3 = 2x_4 = 2\beta$. The first row of the matrix gives the equation $x_1 - x_2 + 2x_4 = 1$, and so $x_1 = 1 + x_2 - 2x_4 = 1 + \alpha - 2\beta$.

Putting all this together, we see the system has the two parameter family of solutions

$$\boxed{(x_1, x_2, x_3, x_4, x_5) = (1 + \alpha - 2\beta, \alpha, 2\beta, \beta, -3)}$$

45 pts.

Problem 2. Consider row operations on matrices with 3 rows.

Recall that for each row operation there is a corresponding elementary matrix E so that EA is the same as the matrix obtained by applying the row operation to A.

(a.) Consider the row operation $R_1 \leftrightarrow R_3$.

i.) Find the corresponding elementary matrix E.

Answer:

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

ii.) Find the inverse row operation.

Answer: $R_1 \leftrightarrow R_3$ (the same operation).

iii.) Find the elementary matrix that corresponds to the inverse row operation.

Answer: The same matrix:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b.) Consider the row operation $R_3 \leftarrow 5R_3$.

i.) Find the corresponding elementary matrix E.

Answer:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

ii.) Find the inverse row operation.

Answer: $R_3 \leftarrow \frac{1}{5}R_3$

iii.) Find the elementary matrix that corresponds to the inverse row operation.

Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

(c.) Consider the row operation $R_2 \leftarrow R_2 + 2R_3$

i.) Find the corresponding elementary matrix E.

Answer:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

ii.) Find the inverse row operation.

Answer: $R_2 \leftarrow R_2 - 2R_3$.

iii.) Find the elementary matrix that corresponds to the inverse row operation.

Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

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40 pts.

Problem 3. In each part, use row operations to determine if the matrix A is invertible and, if so, to find the inverse. It is not necessary to show the individual row operations (you can just use the `rref` key on the calculator). Show the augmented matrix you start with and the augmented matrix you finish with. Give the matrix entries in fractional form.

(a.)

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$$

Answer:

To start, we form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 1 & -5 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

and perform row operations to put the part to the left of the bar in Reduced Row Echelon Form. The result is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & -1 & 1/2 \end{array}\right]$$

Since the reduced row echelon form of A is the identity, we conclude that A is invertible. The inverse is the matrix to the right of the bar, i.e.,

$$A^{-1} = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 0 & 1/2 \\ 2 & -1 & 1/2 \end{bmatrix}.$$

(b.)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

Answer:

We start by forming the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right].$$

We perform row operations to put the part to the left of the bar in RREF. The result is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

Thus, the RREF of A is not the identity, so A is not invertible.

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40 pts.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

(a.) Find the cofactors A_{22} and A_{32}.

Answer:

The cofactor A_{ij} is obtained by crossing out the row i and column j, taking the determinant, and adding the sign from the checkerboard (which can also be described as $(-1)^{i+j}$). Thus,

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = (1)(0-2) = \boxed{-2}$$

and

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = (-1)(0-2) = \boxed{2}$$

(b.) Show how to compute $\det(A)$ by using a cofactor expansion along some row or column.

Answer:

Expand, for example, along column 2, which gives

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} \\ &= 1A_{12} + 1A_{22} + 2A_{32} \\ &= 1\left[-\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}\right] + 1\left[+\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}\right] + 2\left[-\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}\right] \\ &= -(0-2) + (0-2) - 2(0-2) \\ &= 2 - 2 + 4 = \boxed{4} \end{aligned}$$

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40 pts.

Problem 5. **Use row operations** to find the determinant.

$$\begin{vmatrix} 0 & 2 & 6 \\ 1 & 2 & 1 \\ 2 & 6 & 10 \end{vmatrix}$$

Answer:
We proceed as follows:

$$\begin{aligned}
&\begin{vmatrix} 0 & 2 & 6 \\ 1 & 2 & 1 \\ 2 & 6 & 10 \end{vmatrix} && \text{apply } R_1 \leftrightarrow R_2 \\
&= -\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 6 \\ 2 & 6 & 10 \end{vmatrix} && R_3 \leftarrow R_3 - 2R_1 \\
&= -\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 6 \\ 0 & 2 & 8 \end{vmatrix} && R_2 \leftarrow \frac{1}{2}R_2 \\
&= -2\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{vmatrix} && R_3 \leftarrow R_3 - 2R_2 \\
&= -2\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} && \text{now upper triangular, multiply diagonal elements} \\
&= -2(1)(1)(2) \\
&= \boxed{-4}
\end{aligned}$$

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40 pts.

Problem 6. **Use Cramer's rule** to solve the following system.

$$\begin{array}{ccccc} x & + & 3y & = & 2 \\ x & + & y & = & 5 \end{array}$$

Answer:
In matrix form, the system is

$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

By Cramer's rule,

$$x = \frac{\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{-13}{-2} = 13/2$$

and

$$y = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{3}{-2} = -3/2$$

Thus, the solution is

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13/2 \\ -3/2 \end{bmatrix}}.$$
