EXAM

Exam 4
Final Exam

Math 2360–102, Summer I, 2015

July 3, 2015

- Write all of your answers on separate sheets of paper. Do not write on the exam handout. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 8 problems. There are **440 points total**.

Good luck!

40 pts.

Problem 1. In each part, solve the linear system. Use your calculator to find the RREF, but write down the augmented matrix and the matrix you wind up with, and then find all solutions.

A.

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 &= 3 \\ 61x_1 + 60x_2 + 42x_3 &= 56 \\ x_1 + x_2 + x_3 &= 0 \\ 7x_1 + 7x_2 + 6x_3 &= 3 \end{aligned}$$

B.

$$\begin{aligned} 6x_1 + x_2 - 9x_3 + 13x_4 &= -3 \\ -4x_1 - x_2 + 7x_3 - 9x_4 &= 1 \\ 19x_1 + 5x_2 - 34x_3 + 43x_4 &= -4 \\ -13x_1 - 4x_2 + 25x_3 - 30x_4 &= -1 \end{aligned}$$

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40 pts.

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

A. Find the cofactors C_{12}, C_{22} and C_{32}.

B. Show how to compute $\det(A)$, using the cofactor expansion along a selected row or column.

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70 pts.

Problem 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 15 & 29 & -30 & 43 & 4 \\ 9 & 7 & 5 & -14 & 3 & 39 \\ 3 & 2 & 1 & -4 & 0 & 13 \\ 0 & 5 & 10 & -10 & 15 & 0 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A.

D. What is the rank of A?

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60 pts.

Problem 4. You'll want a calculator for this problem. Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -3 \\ -14 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ 1 \end{bmatrix}.$$

Let $S = \text{span}(v_1, v_2, v_3, v_4)$, which is a subspace of $\mathbb{R}^4$.

A. Find a basis of S. What is the dimension of S?

B. Express the vectors in the list $v_1, \ldots, v_4$ that are not part of the basis you found as linear combinations of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 4 \\ 1 \\ -6 \\ 6 \end{bmatrix}.$$

Determine if these vectors are in S. If the vector is in S, express it as a linear combination of the basis vectors found above.

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100 pts.

Problem 5. Let $\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ be the basis of $\mathbb{R}^3$ given by

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let $\mathcal{V}$ be the basis of $\mathbb{R}^3$ given by

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(v_1) = v_1 - 2v_2$$
$$T(v_2) = 3v_1 + v_2$$

A. Find the transition matrices $S_{\mathcal{E}\mathcal{U}}$, $S_{\mathcal{E}\mathcal{V}}$, $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$. Recall that $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$.

B. Find the matrix $[T]_{\mathcal{V}\mathcal{V}}$ of T with respect to $\mathcal{V}$.

C. Find the matrix $[T]_{\mathcal{U}\mathcal{U}}$ of T with respect to $\mathcal{U}$.

D. Let $w \in \mathbb{R}^2$ be the vector with

$$[w]_{\mathcal{U}} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Find $[T(w)]_{\mathcal{U}}$.

E. Find $[T(w)]_{\mathcal{V}}$.

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50 pts.

Problem 6. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$\begin{bmatrix} -3 & 4 & 4 \\ -3 & 4 & 3 \\ -4 & 4 & 4 \end{bmatrix}$$

and the eigenvalues are 1 and 2.

B. The matrix is

$$A = \begin{bmatrix} -3 & 20 \\ -1 & 6 \end{bmatrix}$$

and the eigenvalues are 1 and 2.

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40 pts.

Problem 7. Consider the three vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Apply the Gram-Schmidt Process to these vectors, in the given order, to produce an orthonormal basis of $\mathbb{R}^3$.

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40 pts.

Problem 8. Let S be the subspace of $\mathbb{R}^3$ spanned by the orthonormal vectors

$$u_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}.$$

Let w be the vector

$$w = \begin{bmatrix}2\\1\\3\end{bmatrix}.$$

Find the projection of w onto the subspace S.
