EXAM

Final Exam

Math 2360–202, Summer II, 2014

August 8, 2014

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **430 points total**.

Good luck!

70 pts. **Problem 1**. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 5 \\ 1 & 1 & 0 \end{bmatrix}$$

A. Find the cofactors A_{21} and A_{33}.

B. Compute $\det(A)$, using the cofactor expansion along a selected row or column.

C. Find the adjoint matrix $\operatorname{adj}(A)$ from its definition in terms of cofactors.

D. **Use the information above** to find A^{-1}.

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70 pts. **Problem 2**. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 3 & -1 \\ 2 & 1 & 3 & 1 & 5 & -4 \\ 0 & 2 & -2 & 1 & 0 & -3 \\ 1 & 1 & 1 & 0 & 4 & -2 \\ 2 & 0 & 4 & 1 & 4 & -3 \end{bmatrix}$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A.

D. What is the rank of A?

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70 pts.

Problem 3. You'll want a calculator for this problem. Consider the vectors

$$v_1 = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 18 \\ 10 \\ 1 \\ 4 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Let $S = \operatorname{span}(v_1, v_2, v_3, v_4)$, which is a subspace of $\mathbb{R}^4$.

A. Find a basis of S. What is the dimension of S?

B. Express the vectors in the list $v_1, \dots, v_4$ that are not part of the basis you found as linear combinations of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 8 \\ 7 \\ 4 \\ 6 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -5 \\ 7 \\ 2 \\ 3 \end{bmatrix}.$$

Determine if these vectors are in S. If the vector is in S, express it as a linear combination of the basis vectors found above.

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60 pts.

Problem 4. Let $\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ be the ordered basis of $\mathbb{R}^2$ where

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis $\mathcal{E}$ is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix},$$

in other words, $T(x) = Ax$. Thus, $[T]_{\mathcal{E}\mathcal{E}} = A$.

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

B. Find $[T]_{\mathcal{U}\mathcal{U}}$, the matrix of T with respect to $\mathcal{U}$.

C. **Use the matrix from the last part of the problem** to find the scalars c_1 and c_2 such that

$$T(u_1 - 2u_2) = c_1 u_1 + c_2 u_2.$$

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60 pts.

Problem 5. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

B. The matrix is

$$A = \begin{bmatrix} 11 & 10 & -22 \\ 9 & 11 & -21 \\ 9 & 9 & -19 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

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60 pts.

Problem 6.

A. Consider the three vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Apply the Gram-Schmidt Process to these vectors to produce an orthonormal basis of $\mathbb{R}^3$.

B. Let w be the vector

$$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find the projection of w onto the subspace of $\mathbb{R}^3$ spanned by v_1 and v_2.

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40 pts.

Problem 7. Find the line $y = c_1 x + c_0$ that best fits the following list of data points in the least squares sense,

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\begin{aligned}
(x_1, y_1) &= (1, 21/4)\\
(x_2, y_2) &= (2, 20/3)\\
(x_3, y_3) &= (3, 17/2)\\
(x_4, y_4) &= (4, 19/2).
\end{aligned}
$$

Write the problem in matrix form, write down the normal equations, and solve the normal equations.
