SUMMARY OF LAPLACE TRANSFORMS

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Definition of Laplace Transform: If $f(t)$ is defined for $0 \leq t < \infty$, $\mathscr{L}\{f(t)\} = F(s)$ where

$$F(s) = \int_0^\infty e^{-st} f(x)\, dt.$$

Transforms of Derivatives:

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Translation in s:

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Unit Step Function: The unit step function (a.k.a., the Heaviside function) is

$$\mathscr{U}(t) = \begin{cases} 0, & -\infty < t < 0, \\ 1, & 0 \leq t < \infty. \end{cases}$$

For $a > 0$, we have

$$\mathscr{U}(t-a) = \begin{cases} 0, & -\infty < t < a \\ 1, & a \leq t < \infty. \end{cases}$$

Indicator functions: The indicator function of the interval (a, b) is

$$I_{(a,b)}(t) = \begin{cases} 1, & t \in (a,b) \\ 0, & t \notin (a,b). \end{cases}$$

Assuming that $0 \leq a < b < \infty$ and that t is restricted to $0 \leq t < \infty$, indicator functions can be expressed in terms of unit step functions for the purposes of Laplace transforms by

$$I_{(a,b)}(t) = \mathscr{U}(t-a) - \mathscr{U}(t-b)$$

$$I_{(0,b)}(t) = 1 - \mathscr{U}(t-b)$$

$$I_{(a,\infty)} = \mathscr{U}(t-a).$$

(Given the book's definition of $\mathscr{U}(t-a)$,as above, these are the indicator functions of $[a, b)$, but that doesn't matter for the Laplace transform.)

Translation in t: For $a > 0$,

$$\mathscr{L}\{\mathscr{U}(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\mathscr{L}\{\mathscr{U}(t-a)g(t)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

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Derivatives of Transforms:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Transform of a Periodic Function: If $f(t)$ has period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

Definition of Convolution: If $f(t)$ and $g(t)$ are defined for $0 \leq t < \infty$, then $f * g$ is defined on the same t range by

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau = \int_0^t g(\tau) f(t-\tau)\, d\tau,$$

so $f * g = g * f$.

Convolution Theorem:

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s).$$

Transform of Integrals:

$$\mathscr{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{F(s)}{s}.$$

Transform of the Delta Function:

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}.$$

Impulse response function: Let $P(s)$ be a polynomial of degree n. The solution of the "formal" initial value problem

$$P(D)y = \delta(t), \qquad y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0$$

is

$$h(t) = \mathscr{L}^{-1}\left\{ \frac{1}{P(s)} \right\},$$

which is called the **impulse response function** of the operator $P(D)$. The solution of the initial value problem

$$P(D)y = f(x), \qquad y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0 \tag{$*$}$$

is

$$y(t) = \mathscr{L}^{-1}\{F(s)/P(s)\} = (h * f)(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau.$$

The solution of the homogeneous IVP

$$P(D)y = 0, \qquad y(0) = A_0, y'(0) = A_1, \dots, y^{(n-1)}(0) = A_{n-1}, \tag{$**$}$$

is

$$g(t) = \mathscr{L}^{-1}\{Q(s)/P(s)\}$$

for some polynomial $Q(s)$ of degree $n-1$ that you'd have to calculate. The solution of the IVP

$$P(D)y = f(t), \qquad y(0) = A_0, y'(0) = A_1, \dots, y^{(n-1)}(0) = A_{n-1}$$

is the sum of the solutions to $(*)$ and $(**)$,

$$y(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau + g(t),$$

where (of course) $g(t)$ does not depend of $f(t)$.

Short Table of Laplace Transforms

$f(t)$	$F(s)$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$t\cos(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$\sin(\omega t)-\omega t\cos(\omega t)$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$

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