Geometric Transformations and Wallpaper Groups

Wallpaper Groups

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2010 Math Camp
1 Wallpaper Groups

Introduction
The Structure of Wallpaper Groups
Classification of Lattices
Names for the Wallpaper Groups
How to Classify Wallpaper Groups
Patterns to Classify
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Introduction to Wallpaper Groups

- A **Wallpaper Group** is a discrete group of isometries of the plane that contains noncollinear translations. These are the symmetries of wallpaper patterns.
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Introduction

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The Lattice

• Let $G$ be a wallpaper group.
  • Choose $a \neq 0$ in the lattice $L$ of $G$ so that $\|a\|$ is a small as possible.
  • Choose $b \neq 0$ that is skew to $a$ so that $\|b\|$ is as small as possible.
  • $\|a\| \leq \|b\|$.

Theorem

The lattice $L$ of $G$ is

$$L = \{ma + nb \mid m, n \in \mathbb{Z}\}.$$
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Picture of a Lattice

A Lattice
The Trace

Definition
Here’s some standard linear algebra. If $A$ is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we define $\text{tr}(A)$, the trace of $A$, by

$$\text{tr}(A) = a + d$$

Exercises
All matrices here are $2 \times 2$.

1. If $A$ and $B$ are matrices, show $\text{tr}(AB) = \text{tr}(BA)$. Just use brute force.

2. If $P$ is an invertible matrix, show $\text{tr}(P^{-1}AP) = \text{tr}(A)$. 
The Crystallographic Restriction 1

Theorem (The Crystallographic Restriction)

The possible orders for a rotation in a wallpaper group are 2, 3, 4 and 6.

Proof.

Let $R$ be a rotation in the point group $K(G)$. Then $RL \subseteq L$. This means that $Ra$ must be in the lattice, so $Ra = ma + nb$ for integers $m, n$. Similarly $Rb = pa + qb$ for some $p, q \in \mathbb{Z}$. These equations can be written in matrix form as

$$R[a \mid b] = [Ra \mid Rb] = [a \mid b] \begin{bmatrix} m & p \\ n & q \end{bmatrix}.\]

Let

$$P = [a \mid b], \quad M = \begin{bmatrix} m & p \\ n & q \end{bmatrix}, \quad \text{so } RP = PM.$$
The Crystallographic Restriction 2

Proof Continued.

Since \( a \) and \( b \) are not colinear, \( P = [a \mid b] \) is invertible. so \( R = PMP^{-1} \). But then

\[
\text{tr}(R) = \text{tr}(PMP^{-1}) = \text{tr}(M) = m + q \in \mathbb{Z},
\]

i.e., the trace of \( R \) is an integer. But

\[
R = R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} \implies \text{tr}(R) = 2\cos(\theta)
\]

Thus, \( \cos(\theta) \) must be a half-integer. Since \(-1 \leq \cos(\theta) \leq 1\), the possibilities are \(-1, -1/2, 0, 1/2, 1\).
Crystallographic Restriction 3

Proof Continued.
From Trig, we can figure out the corresponding angles:

\[
\begin{align*}
\cos(\theta) &= -1 \quad \Rightarrow \quad \theta = 180^\circ \quad \Rightarrow \quad o(R) = 2, \\
\cos(\theta) &= -1/2 \quad \Rightarrow \quad \theta = 120^\circ \quad \Rightarrow \quad o(R) = 3, \\
\cos(\theta) &= 0 \quad \Rightarrow \quad \theta = 90^\circ \quad \Rightarrow \quad o(R) = 4, \\
\cos(\theta) &= 1/2 \quad \Rightarrow \quad \theta = 60^\circ \quad \Rightarrow \quad o(R) = 6, \\
\cos(\theta) &= 1 \quad \Rightarrow \quad \theta = 0 \quad \Rightarrow \quad R = I.
\end{align*}
\]
Picture of the Angles

- Here's the picture of the angles.

- 
  \[ (\frac{-1}{2}, \frac{\sqrt{3}}{2}) \]
  \[ (\frac{0}{2}, \frac{\sqrt{3}}{2}) \]
  \[ (0, 1) \]
  \[ (\frac{1}{2}, \frac{\sqrt{3}}{2}) \]

- Angles:
  - \( 0^\circ \)
  - \( 60^\circ \)
  - \( 90^\circ \)
  - \( 120^\circ \)
  - \( 180^\circ \)
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Classification of Lattices 1

- The lattices of wallpaper groups can be divided into 5 classes.
  - Since $a \pm b$ are skew to $a$, $\|b\| \leq \|a + b\|, \|a - b\|$.
  - We can arrange $\|a - b\| \leq \|a + b\|$ by replacing $b$ by $-b$, if necessary. Then
    $$\|a\| \leq \|b\| \leq \|a - b\| \leq \|a + b\|.$$  
  - We get 8 cases by considering $\leq$ or $<$ for each of the $\leq$'s above.
Classification of Lattices 2

<table>
<thead>
<tr>
<th>Case</th>
<th>Inequality</th>
<th>Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>3</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>4</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>5</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>6</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>7</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>8</td>
<td>$</td>
<td>a</td>
</tr>
</tbody>
</table>

- Let’s look at the pictures.
Case 8, Oblique Lattice

\[ \begin{array}{ccc}
  & b \\
a & & \\
\end{array} \]
Case 8, Oblique Lattice
Case 7, Rectangular Lattice
Case 7, Rectangular Lattice
Case 3, Square Lattice
Case 2, Hexagonal Lattice
Case 2, Hexagonal Lattice
Case 2, Hexagonal Lattice
Case 6, Centered Rectangular
Case 6, Centered Rectangular
Case 4, Another Centered Rectangular
Case 4, Another Centered Rectangular
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Theorem

*The are 17 geometric isomorphism classes of wallpaper groups.*

- **Conway notation**
  - 2, 3, 4, 6 indicate a class of rotocenters of the given order.
  - * indicates a mirror. Digits after the star mean the rotocenters are on a mirror line.
  - x indicates a glide reflection (that does not come from a pure reflection).
  - 1 means there are only translations in the group.

- **Crystallographic Notation (abbreviations from a longer, logical system)**
  - p or c mean a primitive or centered cell in the lattice.
  - 2, 3, 4, 6 denotes the highest order of a rotation in the group.
  - m for mirror, stands for reflection, g stands for a glide.
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Flowchart for Classification
Outline

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Patterns to Classify

• Classify the following patterns. Don’t look at the answers until you’ve given it a try!
Wallpaper Problem 1
Wallpaper Problem 2
Wallpaper Problem 3
Wallpaper Problem 4
Wallpaper Problem 5
End of Lectures
Answer to Wallpaper Problem 1

*632
Answer to Wallpaper Problem 2

3 * 3
Answer to Wallpaper Problem 3

*2222
Answer to Wallpaper Problem 4

22*
Answer to Wallpaper Problem 5

*333*