Geometric Transformations and Wallpaper Groups

Symmetries of Geometric Patterns
(Discrete Groups of Isometries)

Lance Drager

Department of Mathematics and Statistics
Texas Tech University
Lubbock, Texas

2010 Math Camp
1. Discrete Groups of Isometries
   - What Makes a Group Discrete?
   - The Point Group
   - Three Kinds of Groups
   - Classification of Rosette Groups
   - The Lattice and the Point Group

2. Frieze Groups
   - Definition and Lattice
   - Geometric Isomorphism
   - Names for the Frieze Groups
   - The 7 Frieze Groups
   - Classification Flowchart
Discrete Groups

• Let $G \subseteq \mathbb{E}$ be a group of isometries. If $p$ is a point, the orbit of $p$ under $G$ is

$$\text{Orb}(p) = \{gp \mid g \in G\}.$$ 

Definition
A group $G$ of isometries is discrete if, for every orbit $\Gamma$ of a point under $G$, there is a $\delta > 0$ so that

$$p, q \in \Gamma \text{ and } p \neq q \implies \text{dist}(p, q) \geq \delta.$$ 

• $D_3$, as the symmetries of an equilateral triangle, is discrete. The orthogonal group $O$ is not discrete.
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The Point Group

Definitions

Let $G$ be a discrete group of isometries. We can define a group map

$$\pi : G \to \mathcal{O} : (A | a) \mapsto A.$$  

The kernel $N(G)$ of this map is the normal subgroup of $G$ consisting of all the translations in $G$. Naturally, it’s called the translation subgroup of $G$.

The subgroup $K(G) = \pi(G) \subseteq \mathcal{O}$ is called the Point Group of $G$. Note carefully that the point group $K(G)$ of $G$ is not a subgroup of $G$!
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Three Kinds of Groups

- There are three possibilities for the translation subgroup $N(G)$ of $G$.
  1. $N(G)$ contains only the identity. These groups $G$ are the symmetries of rosette patterns and are called rosette groups.
  2. All the translations in $N(G)$ are collinear. These groups are the symmetry groups of frieze patterns and are called frieze groups.
  3. $N(G)$ contains noncollinear translations. These groups are the symmetries of wallpaper patterns and are called wallpaper groups.

- We’ll describe the classification for each of these types of groups.
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Classification of Rosette Groups

- Rosette groups are the symmetries of rosette patterns
Cyclic Groups

- A group is **cyclic** if it consists of the powers of a single element. The **order of a group** is the number of elements in the group.

- An element $g \in G$ has **order $n$** if $n$ is the smallest number so that $g^n = e$. (If there’s no such $n$, then $g$ has infinite order). In this case the cyclic group $\langle g \rangle$ has $n$ elements,

  $$\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}.$$  

- A cyclic group of order $n$ is isomorphic to the group $C_n$ consisting of the integers

  $$0, 1, 2, \ldots, n-1$$  

  with addition mod $n$ (clock arithmetic).
Classification of Rosette Groups

Theorem (Classification of Rosette Groups)

A rosette group $G$ is either a cyclic group $C_n$ generated by a rotation or is one of the dihedral groups $D_n$.

Outline of Proof.

1. There is smallest rotation $r$ in $G$, which has finite order $n$. This takes care of all the rotations, so the rotation subgroup is $\langle r \rangle \approx C_n$.

2. If there is no reflection, we’re done. Otherwise there is a reflection $s = S(\theta)$ with a smallest $\theta$.

3. If there are no rotations, $G = \langle s \rangle$, which is $D_1$.

4. If $r$ has order $n$, we have the reflections $s, rs, \ldots r^{n-1}s$ and no others. The group is $D_n$. 

Show me the proof
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The Lattice and the Point Group

Definition
If $G$ is a discrete group of isometries, the lattice $L$ of $G$ is the orbit of the origin under the translation subgroup $N(G)$ of $G$. To put it another way,

$$L = \{v \in \mathbb{R}^2 \mid (I|v) \in G\}$$

- Observe that if $a, b \in L$ then $ma + nb \in L$ for all integers $m$ and $n$. This is because $(I|a)$ and $(I|b)$ are in $G$, so $(I|a)^m = (I|ma)$ and $(I|b)^n = (I|nb)$ are in $G$. Thus $ma + nb = (I|ma)(I|nb)0$ is in $L$. 
The Point Group Preserves the Lattice

**Theorem**

Let $L$ be the lattice of $G$ and let $K = K(G)$ be the point group of $G$. Then $KL \subseteq L$, i.e., if $A \in K$ and $v \in L$ then $Av \in L$.

**Proof.**

We must have $(I | v) \in G$ and $(A | a) \in G$ for some $a$. Then

$$(A | a)(I | v)(A | a)^{-1} = (A | a + Av)(A^{-1} | - A^{-1}a)$$

$$= (AA^{-1} | a + Av + A(-A^{-1}a))$$

$$= (I | Av).$$

Thus, $Av \in L$.  $\square$
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   Classification Flowchart
A discrete group of isometries $G$ is a **frieze group** if all the vectors in the lattice are collinear. These are the symmetry groups of frieze patterns.
Frieze Groups 2

- A wallpaper border.
The Lattice

**Theorem**

Let $G$ be a frieze group and let $L$ be its lattice. Choose $a \in L$ so that $0 < \|a\|$ is as small as possible. Such an $a$ exists because the group is discrete. Then

$$L = \{na \mid n \in \mathbb{Z}\}.$$

**Proof.**

Suppose not. Then there is a $c \in L$ so that $c \neq na$ for an $n \in \mathbb{Z}$. But must have $c = sa$ for some scalar $s$, so $s \notin \mathbb{Z}$. Then we have $s = k + r$ where $k$ is an integer and $0 < r < 1$. But then $c = ka + ra$. Since $c, ka \in L$, this implies $ra = c - ka \in L$. But $0 < \|ra\| = |r| \|a\| < \|a\|$, which contradicts the choice of $a$. \qed
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**Definition**

Two discrete groups of isometries $G_1$ and $G_2$ are geometrically isomorphic if there is a group isomorphism $\phi: G_1 \rightarrow G_2$ that preserves the geometric type of the elements, i.e., $\phi(T)$ always has the same type as $T$.

- We classify frieze (and wallpaper groups) up to geometric isomorphism.
- If $T \in \mathbb{E}$, the conjugation map $\psi_T: \mathbb{E} \rightarrow \mathbb{E}: Q \mapsto TQT^{-1}$ preserves type. Hence, for any subgroup $G$, $\psi_T(G)$ and $G$ are geometrically isomorphic.
Lattice Direction and Classification

- If $G$ is a frieze group, we can conjugate $G$ by a rotation, to get a geometrically isomorphic group where $a$ points in the positive $x$-direction. Thus, the lattice $L$ looks like this.

- There are only four possible elements of the point group.
  1. The identity
  2. Vertical reflection, i.e., reflection through the $y$-axis
  3. Horizontal reflection, i.e., reflection through the $x$-axis.
  4. A half-turn ($180^\circ$) rotation.

Theorem

There are exactly seven geometric isomorphism classes of frieze groups.
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Names for the Frieze Groups

- A system for naming these groups has been standardized by crystallographers. Another (better) naming system has been invented by Conway, who also invented English names for these classes.

- The crystallographic names consist of two symbols.
  1. **First Symbol**
     1. No vertical reflection.
     \( m \) A vertical reflection.
  2. **Second Symbol**
     1. No other symmetry.
     \( m \) Horizontal reflection.
     \( g \) Glide reflection (horizontal).
     2. Half-turn rotation.
Conway Names

- In the Conway system two rotations are of the same class if their rotocenters differ by a motion in the group. Two reflections are of the same class if their mirror lines differ by a motion in the group.

- Conway Names
  - $\infty$ We think of the translations as “rotation” about a center infinitely far away. There are two rotocenters, one in the up direction, and one in the down direction. These “rotations” have order $\infty$.
  - 2 A class of half-turn rotations.
  - $\ast$ Shows the presence of a reflection. If a 2 or $\infty$ comes after the $\ast$, then the rotocenters are at the intersection of mirror lines.
  - $\times$ Indicates the presence of a glide reflection.
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The 7 Frieze Groups

Hop, 11, ∞∞
The 7 Frieze Groups 1

**Hop, 11, ∞∞**

**Jump, 1m, ∞***
The 7 Frieze Groups

Sidle, \( m1, \infty \infty \)
The 7 Frieze Groups

Sidle, \( m1, \ast \infty \infty \)

Step, \( 1g, \infty \times \)
The 7 Frieze Groups 3

Spinhop, 12, 22∞
The 7 Frieze Groups 3

Spinhop, $12, 22\infty$

Spijnump, $mm, \ast 22\infty$
The 7 Frieze Groups 4

Spinsidle, $mg$, $2 \ast \infty$
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How to Classify Frieze Groups

Vertical Reflection?
- yes
  - Horizontal Reflection?
    - yes
      - Half-Turn?
        - yes
          - mg spin-sidle 2*∞
        - no
          - m1 sidle *∞∞
    - no
      - Horizontal Reflection or Glide?
        - yes
          - Horizontal Reflection?
            - yes
              - Half-Turn?
                - yes
                  - 12 spinhop 22∞
                - no
                  - 11 hop ∞∞
            - no
              - 1m jump ∞*
        - no
          - 1g step ∞x

mm spin-jump *22∞
Proof of the Classification of Rosette Groups

- Let $G$ be a rosette group. Let $J$ be the subgroup of $G$ consisting of rotations.
- Suppose $J \neq \{I\}$.
- There is a smallest $\theta$ in the range $0 < \theta < 360$ so that $r = R(\theta) \in G$, since otherwise the group is not discrete.
- Claim: There is an $n \in \mathbb{N}$ so that $n\theta = 360$.
  - Suppose not. Then there is an integer $k$ so that $k\theta < 360 < (k + 1)\theta$, so $360 = k\theta + \phi$, where $0 < \phi < \theta$.
  - This means $I = R(360) = R(k\theta + \phi) = R(\theta)^k R(\phi)$.
  - But then $R(\phi) = R(\theta)^{-k} \in G$, where $0 < \phi < \theta$, which contradicts the definition of $\theta$. 
Proof of the Classification of Rosette Groups 2

• **Claim:** \( J = \{I, r, r^2, \ldots, r^{n-1}\} \).

  • Suppose not. Then there is some \( R(\phi) \in J \), where \( 0 < \phi < 360 \) and \( \phi \) is not an integer multiple of \( \theta \). As above, we can find an integer \( k \) so that \( k\theta < \phi < (k+1)\theta \), so \( \phi = k\theta + \psi \), where \( 0 < \psi < \theta \).

  • But then \( R(\phi) = R(k\theta + \psi) = r^kR(\psi) \) is in \( G \), so \( R(\psi) = r^{-k}R(\phi) \in G \). Since \( 0 < \psi < \theta \), this contradicts the definition of \( \theta \).

• **Conclusion:** \( J \) is a finite cyclic group.

• **Consider Reflections in \( G \).** If there a no reflections, we’re done, so suppose there is at least one reflection. There is a smallest \( \phi \) is the range \( 0 \leq \phi < 360 \) so that \( S(\phi) \) is in \( G \). Otherwise the group is not discrete. Let \( s = S(\phi) \).
Proof of the Classification of Rosette Groups 3

- **Case 1**: \( J = \{ I \} \).
- **Claim**: \( G = \{ I, s \} \), which is \( D_1 \).
  - Suppose not. There must be another reflection \( S(\psi) \) in \( G \), \( \phi < \psi < 360 \). But then \( G \) contains \( S(\psi)S(\phi) = R(\psi - \phi) \), which is a rotation that is not the identity, contradicting our assumption on \( J \).
- **Case 2**: \( J = \{ I, r, r^2, \ldots, r^{n-1} \} \).
  - \( G \) must contain the reflections \( s, rs, \ldots, r^{n-1}s \). These are the reflections \( r^k s = R(\theta)^k S(\phi) = S(k\theta + \phi) \), where \( k \in \{0, 1, \ldots, n - 1\} \).
Proof of the Classification of Rosette Groups 4

- **Claim:** There are no other reflections in $G$.
  - Suppose not. There is some $S(\psi) \in G$ where $\phi < \psi < 360$, but $\psi \neq k\theta + \phi$ for any $k \in \{0, 1, \ldots, n - 1\}$.
  - But then $G$ contains $S(\psi)S(\phi) = R(\psi - \phi)$. Since $0 < \psi - \phi < 360$ we must have $\psi - \phi = k\theta$ for some $k \in \{1, 2, \ldots, n - 1\}$. This is a contradiction.

- We now know that $G = \{1, r, r^2, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$.
  - For any angles $\alpha$ and $\beta$, $S(\alpha)R(\beta)S(\alpha) = R(-\beta) = R(\beta)^{-1}$.
  - Thus, $srs = r^{-1} = r^{n-1}$, so $sr = r^{n-1}s$. Thus, $G = D_n$.
  - This finishes Case 2.

- The proof is complete!