Geometric Transformations and Wallpaper Groups

Symmetries of Geometric Patterns (Discrete Groups of Isometries)

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Outline

Discrete Groups

- Let $G \subseteq \mathbb{E}$ be a group of isometries. If p is a point, the orbit of p under G is

$$\operatorname{Orb}(p) = \{gp \mid g \in G\}.$$

Definition

A group G of isometries is discrete if, for every orbit Γ of a point under G, there is a $\delta > 0$ so that

$$p, q \in \Gamma \text{ and } p \neq q \implies \operatorname{dist}(p, q) \geq \delta.$$

- D_3, as the symmetries of an equilateral triangle, is discrete. The orthogonal group $\mathbb{O}$ is not discrete.

The Point Group

Definitions

Let G be a discrete group of isometries. We can define a group map

$$\pi \colon G \to \mathbb{O} \colon (A \mid a) \mapsto A.$$

The kernel $N(G)$ of this map is the normal subgroup of G consisting of all the translations in G. Naturally, it's called the translation subgroup of G.
The subgroup $K(G) = \pi(G) \subseteq \mathbb{O}$ is called the Point Group of G. Note carefully that the point group $K(G)$ of G is not a subgroup of G!

Three Kinds of Groups

- There are three possibilities for the translation subgroup $N(G)$ of G.
  1. $N(G)$ contains only the identity. These groups G are the symmetries of rosette patterns and are called rosette groups.
  2. All the translations in $N(G)$ are collinear. These groups are the symmetry groups of frieze patterns and are called frieze groups.
  3. $N(G)$ contains noncollinear translations. These groups are the symmetries of wallpaper patterns and are called wallpaper groups.
- We'll describe the classification for each of these types of groups.

Classification of Rosette Groups

- Rosette groups are the symmetries of rosette patterns

Cyclic Groups

- A group is cyclic if it consists of the powers of a single element. The order of a group is the number of elements in the group.
- An element $g \in G$ has order n if n is the smallest number so that $g^n = e$. (If there's no such n, then g has infinite order). In this case the cyclic group $\langle g \rangle$ has n elements,

$$\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}.$$

- A cyclic group of order n is isomorphic to the group C_n consisting of the integers

$$0, 1, 2, \ldots, n-1$$

with addition mod n (clock arithmetic).

Classification of Rosette Groups

Theorem (Classification of Rosette Groups)

A rosette group G is either a cyclic group C_n generated by a rotation or is one of the dihedral groups D_n.

Outline of Proof.

1. There is smallest rotation r in G, which has finite order n. This takes care of all the rotations, so the rotation subgroup is $\langle r \rangle \approx C_n$.
2. If there is no reflection, we're done. Otherwise there is a reflection $s = S(\theta)$ with a smallest θ.
3. If there are no rotations, $G = \langle s \rangle$, which is D_1
4. If r has order n, we have the reflections $s, rs, \ldots r^{n-1}s$ and no others. The group is D_n.

Show me the proof

The Lattice and the Point Group

Definition

If G is a discrete group of isometries, the lattice L of G is the orbit of the origin under the translation subgroup $N(G)$ of G. To put it another way,

$$L = \{v \in \mathbb{R}^2 \mid (I \mid v) \in G\}$$

- Observe that if $a, b \in L$ then $ma + nb \in L$ for all integers m and n. This is because $(I \mid a)$ and $(I \mid b)$ are in G, so $(I \mid a)^m = (I \mid ma)$ and $(I \mid b)^n = (I \mid nb)$ are in G. Thus $ma + nb = (I \mid ma)(I \mid nb)0$ is in L.

The Point Group Preserves the Lattice

Theorem

Let L be the lattice of G and let $K = K(G)$ be the point group of G. Then $KL \subseteq L$, i.e., if $A \in K$ and $v \in L$ then $Av \in L$.

Proof.

We must have $(I \mid v) \in G$ and $(A \mid a) \in G$ for some a. Then

$$\begin{aligned}(A \mid a)(I \mid v)(A \mid a)^{-1} &= (A \mid a + Av)(A^{-1} \mid -A^{-1}a)\\ &= (AA^{-1} \mid a + Av + A(-A^{-1}a))\\ &= (I \mid Av).\end{aligned}$$

Thus, $Av \in L$. □

Frieze Groups 1

- A discrete group of isometries G is a frieze group if all the vectors in the lattice are collinear. These are the symmetry groups of frieze patterns.

Frieze Groups 2

- A wallpaper border.

The Lattice

Theorem

Let G be a frieze group and let L be its lattice. Choose $a \in L$ so that $0 < \|a\|$ is as small as possible. Such an a exists because the group is discrete. Then

$$L = \{na \mid n \in \mathbb{Z}\}.$$

Proof.

Suppose not. Then there is a $c \in L$ so than $c \neq na$ for an $n \in \mathbb{Z}$. But must have $c = sa$ for some scalar s, so $s \notin \mathbb{Z}$. Then we have $s = k + r$ where k is an integer and $0 < r < 1$. But then $c = ka + ra$. Since $c, ka \in L$, this implies $ra = c - ka \in L$. But $0 < \|ra\| = |r| \|a\| < \|a\|$, which contradicts the choice of a. □

Geometric Isomorphism

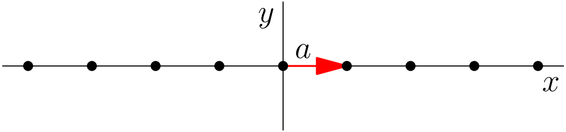
Definition

Two discrete groups of isometries G_1 and G_2 are geometrically isomorphic if the is a group isomorphism $\phi\colon G_1 \to G_2$ that preserves the geometric type of the elements, i.e., $\phi(T)$ always has the same type as T.

- We classify frieze (and wallpaper groups) up to geometric isomorphism
- If $T \in \mathbb{E}$, the conjugation map $\psi_T\colon \mathbb{E} \to \mathbb{E}\colon Q \mapsto TQT^{-1}$ preserves type. Hence, for any subgroup G, $\psi_T(G)$ and G are geometrically isomorphic.

Lattice Direction and Classification

- If G is a frieze group, we can conjugate G by a rotation, to get a geometrically isomorphic group where a points in the positive x-direction. Thus, the lattice L looks like this.



- There are only four possible elements of the point group.
  1. The identity
  2. Vertical reflection, i.e., reflection through the y-axis
  3. Horizontal reflection, i.e, reflection through the x-axis.
  4. A half-turn (180°) rotation.

Theorem

There are exactly seven geometric isomorphism classes of frieze groups.

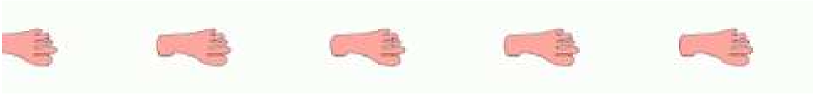
Names for the Frieze Groups

- A system for naming these groups has been standardized by crystallographers. Another (better) naming system has been invented by Conway, who also invented English names for these classes.
- The crystallographic names consist of two symbols.
  1. First Symbol
     - 1 No vertical reflection.
     - m A vertical reflection.
  2. Second Symbol
     - 1 No other symmetry.
     - m Horizontal reflection.
     - g Glide reflection (horizontal).
     - 2 Half-turn rotation.

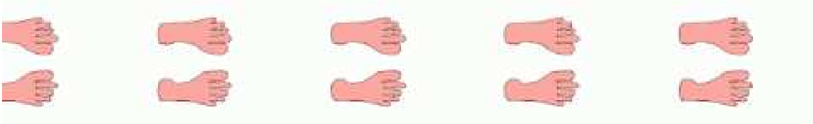
Conway Names

- In the Conway system two rotations are of the same class if their rotocenters differ by a motion in the group. Two reflections are of the same class if their mirror lines differ by a motion in the group.
- Conway Names
  - ∞ We think of the translations as "rotation" about a center infinitely far away. There are two rotocenters, one in the up direction, and one in the down direction. These "rotations" have order ∞.
  - 2 A class of half-turn rotations.
  - $*$ Shows the presence of a reflection. If a 2 or ∞ comes after the $*$, then the rotocenters are at the intersection of mirror lines.
  - x Indicates the presence of a glide reflection.

The 7 Frieze Groups 1



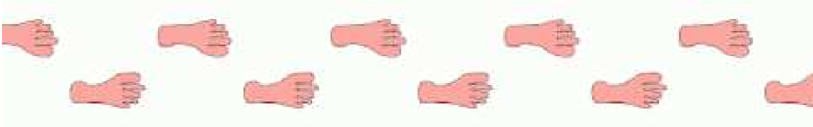
Hop, 11, $\infty\infty$



Jump, 1m, $\infty *$

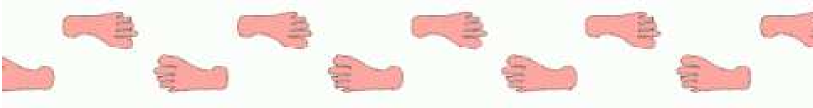
The 7 Frieze Groups 2

Sidle, $m1$, $*\infty\infty$

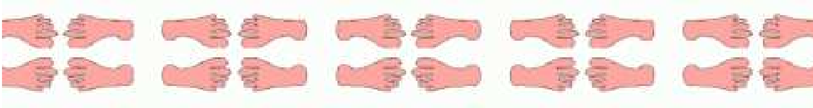


Step, $1g$, ∞x

The 7 Frieze Groups 3

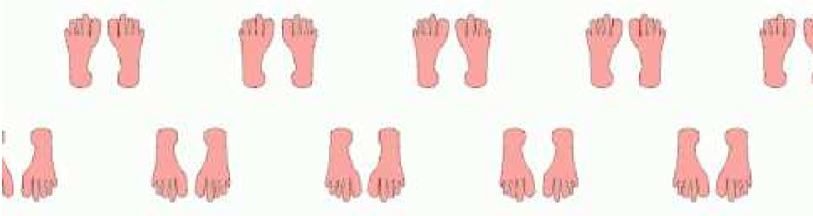


Spinhop, 12, 22∞



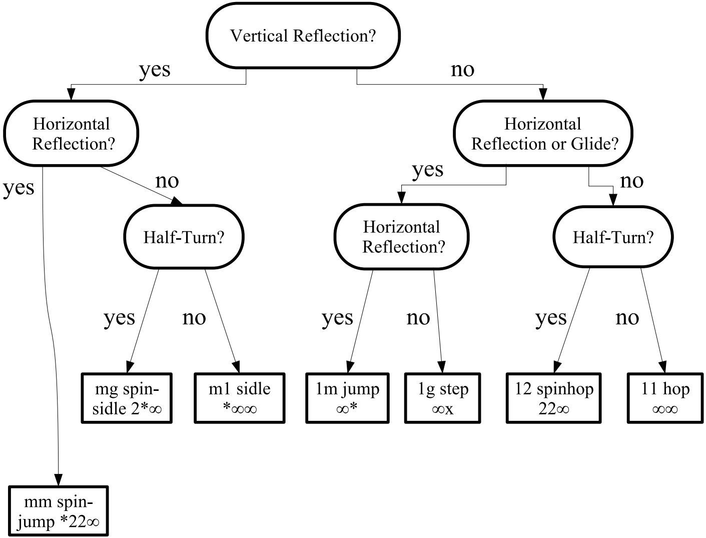
Spinjump, *mm*, $*22\infty$

The 7 Frieze Groups 4



Spinsidle, mg, $2 * \infty$

How to Classify Frieze Groups



Proof of the Classification of Rosette Groups 1

- Let G be a rosette group. Let J be the subgroup of G consisting of rotations.
- Suppose $J \neq \{I\}$.
- There is a smallest θ in the range $0 < \theta < 360$ so that $r = R(\theta) \in G$, since otherwise the group is not discrete.
- Claim: There is an $n \in \mathbb{N}$ so that $n\theta = 360$.
  - Suppose not. Then there is an integer k so that $k\theta < 360 < (k+1)\theta$, so $360 = k\theta + \phi$, where $0 < \phi < \theta$.
  - This means $I = R(360) = R(k\theta + \phi) = R(\theta)^k R(\phi)$.
  - But then $R(\phi) = R(\theta)^{-k} \in G$, where $0 < \phi < \theta$, which contradicts the definition of θ.

Proof of the Classification of Rosette Groups 2

- Claim: $J = \{I, r, r^2, \ldots, r^{n-1}\}$.
  - Suppose not. Then there is some $R(\phi) \in J$, where $0 < \phi < 360$ and ϕ is not an integer multiple of θ. As above, we can find an integer k so that $k\theta < \phi < (k+1)\theta$, so $\phi = k\theta + \psi$, where $0 < \psi < \theta$.
  - But then $R(\phi) = R(k\theta + \psi) = r^k R(\psi)$ is in G, so $R(\psi) = r^{-k} R(\phi) \in G$. Since $0 < \psi < \theta$, this contradicts the definition of θ.
- Conclusion: J is a finite cyclic group.
- Consider Reflections in G. If there a no reflections, we're done, so suppose there is at least one reflection. There is a smallest ϕ is the range $0 \leq \phi < 360$ so that $S(\phi)$ is in G. Otherwise the group is not discrete. Let $s = S(\phi)$.

Proof of the Classification of Rosette Groups 3

- Case 1: $J = \{I\}$.
- Claim: $G = \{I, s\}$, which is D_1.
  - Suppose not. There must be another reflection $S(\psi)$ in G, $\phi < \psi < 360$. But then G contains $S(\psi)S(\phi) = R(\psi - \phi)$, which is a rotation that is not the identity, contradicting our assumption on J.
- Case 2: $J = \{I, r, r^2, \ldots, r^{n-1}\}$.
- G must contain the reflections $s, rs, \ldots, r^{n-1}s$. These are the reflections $r^k s = R(\theta)^k S(\phi) = S(k\theta + \phi)$, where $k \in \{0, 1, \ldots, n-1\}$.

Proof of the Classification of Rosette Groups 4

- Claim: There are no other reflections in G.
  - Suppose not. There is some $S(\psi) \in G$ where $\phi < \psi < 360$, but $\psi \neq k\theta + \phi$ for any $k \in \{0, 1, \dots, n-1\}$.
  - But then G contains $S(\psi)S(\phi) = R(\psi - \phi)$. Since $0 < \psi - \phi < 360$ we must have $\psi - \phi = k\theta$ for some $k \in \{1, 2, \dots, n-1\}$. This is a contradiction.
- We now know that $G = \{I, r, r^2, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s\}$.
- For any angles α and β, $S(\alpha)R(\beta)S(\alpha) = R(-\beta) = R(\beta)^{-1}$.
- Thus, $srs = r^{-1} = r^{n-1}$, so $sr = r^{n-1}s$. Thus, $G = D_n$.
- This finishes Case 2.
- The proof is complete! Back to the Theorem