Geometric Transformations and Wallpaper Groups

Isometries of the Plane

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Isometries

- A transformation $T$ of the plane is an **isometry** if it is one-to-one and onto and
  \[
  \text{dist}(T(p), T(q)) = \text{dist}(p, q), \quad \text{for all points } p \text{ and } q.
  \]

- If $S$ and $T$ are isometries, so is $ST$, where $(ST)(p) = S(T(p))$.

- If $T$ is an isometry, so is $T^{-1}$.

- Our goal is to find all the isometries.
Orthogonal Matrices

- A matrix $A$ is **orthogonal** if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^2$.
- An orthogonal matrix gives an isometry of the plane since

$$\text{dist}(Ap, Aq) = \|Ap - Aq\|$$

$$= \|A(p - q)\| = \|p - q\| = \text{dist}(p, q)$$

- If $A$ and $B$ are orthogonal, so is $AB$.
- If $A$ is orthogonal, it is invertible and $A^{-1}$ is orthogonal.
- Rotation matrices are orthogonal.
Dot Product 1

**Theorem**

A is orthogonal if and only if $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^2$. Thus, an orthogonal matrix preserves angles.

**Proof.**

$(\implies)$

$$\|Ax\|^2 = Ax \cdot Ax = x \cdot x = \|x\|^2$$

$(\iff)$ Recall

$$2x \cdot y = \|x\|^2 + \|y\|^2 - \|x - y\|^2.$$
Proof Continued.

So, we have

$$2Ax \cdot Ay = \|Ax\|^2 + \|Ay\|^2 - \|Ax - Ay\|^2$$

$$= \|Ax\|^2 + \|Ay\|^2 - \|A(x - y)\|^2 \quad \text{(distributive law)}$$

$$= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \quad \text{(A is orthogonal)}$$

$$= 2x \cdot y,$$

and dividing by 2 gives the result.

**Theorem**

*A matrix is orthogonal if and only if its columns are orthogonal unit vectors.*
Proof.

\((\implies\implies)\)

\[ \|Ae_i\| = \|e_i\| = 1. \]

\[ Ae_1 \cdot Ae_2 = e_1 \cdot e_2 = 0. \]

\((\iff)\) Let \( A = [u \mid v] \) where \( u \) and \( v \) are orthogonal unit vectors. Note \( Ax = x_1 u + x_2 v. \)

\[ \|Ax\|^2 = Ax \cdot Ax \]

\[ = (x_1 u + x_2 v) \cdot (x_1 u + x_2 v) \]

\[ = x_1^2 u \cdot u + 2x_1x_2 u \cdot v + x_2^2 v \cdot v \]

\[ = x_1^2 (1) + 2x_1x_2 (0) + x_2^2 (1) \]

\[ = x_1^2 + x_2^2 = \|x\|^2 \]
Classifying Orthogonal Matrices 1

- If $A$ is orthogonal, $\text{Col}_1(A) = (\cos(\theta), \sin(\theta))$ for some $\theta$. So the possibilities are

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta),$$

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = S(\theta).$$

- In the first case $A = R(\theta)$ is a rotation.
- What about $S(\theta)$?
- There are orthogonal unit vectors $u$ and $v$ so that $S(\theta)u = u$ and $S(\theta)v = -v$. 
In fact, let \( u = (\cos(\theta/2), \sin(\theta/2)) \) and
\( v = (-\sin(\theta/2), \cos(\theta/2)). \)

\[
S(\theta)u = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\theta/2) + \sin(\theta)\sin(\theta/2) \\ \sin(\theta)\cos(\theta/2) - \cos(\theta)\sin(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta - \theta/2) \\ \sin(\theta - \theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = u.
\]
Classifying Orthogonal Matrices 3

$S(\theta)v = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
\sin(\theta) & -\cos(\theta)
\end{bmatrix} \begin{bmatrix}
-\sin(\theta/2) \\
\cos(\theta/2)
\end{bmatrix}$

$= \begin{bmatrix}
-\cos(\theta)\sin(\theta/2) + \sin(\theta)\cos(\theta/2) \\
-\sin(\theta)\sin(\theta/2) - \cos(\theta)\cos(\theta/2)
\end{bmatrix}$

$= \begin{bmatrix}
\sin(\theta - \theta/2) \\
-\cos(\theta - \theta/2)
\end{bmatrix}$

$= \begin{bmatrix}
\sin(\theta/2) \\
-\cos(\theta/2)
\end{bmatrix} = -\begin{bmatrix}
-\sin(\theta/2) \\
\cos(\theta/2)
\end{bmatrix} = -v.$
Classifying Orthogonal Matrices 4

- $S(\theta)u = u$ and $S(\theta)v = -v$. If $x = tu + sv$, then $S(\theta)x = tu - sv$.

- $S(\theta)$ is a reflection with its mirror line at an angle of $\theta/2$. 
Exercises

• $S(\theta)^2 = I$, so $S(\theta)^{-1} = S(\theta)$.

• Let $A$ be an orthogonal matrix. Then $A$ is a rotation (or the identity) if and only if $\det(A) = 1$ and $A$ is a reflection if and only if $\det(A) = -1$.

• If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ define the transpose of $A$ by $A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $A$ is orthogonal if and only if $A^{-1} = A^T$. 
Exercises on Orthogonal Matrices

Exercises Continued

• Use the addition laws for sine and cosine to verify the important identities

\[ R(\theta)S(\phi) = S(\theta + \phi), \]
\[ S(\phi)R(\theta) = S(\phi - \theta), \]
\[ S(\theta)S(\phi) = R(\theta - \phi) \]
Translations

- For $v \in \mathbb{R}^2$, define $T_v : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_v(x) = x + v$. We say $T_v$ is translation by $v$.
- $T_u T_v = T_{u+v}$ and $T_v^{-1} = T_{-v}$.
- Translations are isometries

$$\text{dist}(T_v(x), T_v(y)) = \|T_v(x) - T_v(y)\|$$
$$= \|(x + v) - (y + v)\|$$
$$= \|x + v - y - v\|$$
$$= \|x - y\|$$
$$= \text{dist}(x, y).$$
Three Distances Determine a Point

- Let $p_1$, $p_2$ and $p_3$ be noncolinear points. A point $x$ is uniquely determined by the three numbers $r_1 = \text{dist}(x, p_1)$, $r_2 = \text{dist}(x, p_2)$ and $r_3 = \text{dist}(x, p_3)$. 
First Classification

- If an isometry $T$ fixes three noncolinear points $p_1, p_2, p_3$ then $T$ is the identity.

$$\text{dist}(x, p_i) = \text{dist}(T(x), T(p_i)) = \text{dist}(T(x), p_i), \quad i = 1, 2, 3,$$

$$\implies x = T(x).$$

**Theorem**

*Every isometry $T$ can be written as $T(x) = Ax + v$ where $A$ is an orthogonal matrix, i.e., $T$ is multiplication by an orthogonal matrix followed by a translation.*
Proof of First Classification

Proof.

1. The points $0, e_1, e_2$ are noncolinear.
2. Let $u = -T(0)$. Then $T_u T(0) = 0$.
3. $\text{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix $R$ so that $R T_u T(e_1) = e_1$.
4. $R T_u T(e_2) = \pm e_2$.
5. If $R T_u T(e_2) = -e_2$, let $S = S(0)$ be reflection through the $x$-axis, otherwise let $S = I$.
6. $S R T_u(0) = 0$, $S R T_u T(e_1) = e_2$, and $S R T_u T(e_2) = e_2$
7. $S R T_u T = I$, so $T = T_{-u} R^{-1} S^{-1}$
8. Let $A = R^{-1} S^{-1}$ and $v = -u$. Then $T(x) = Ax + v$ \(\square\)
Notation and Algebra

- If $T(x) = Ax + v$ write $T = (A | v)$.
- Calculate the product (composition) in this notation

\[
(A | u)(B | v)x = (A | u)(Bx + v) \\
= A(Bx + v) + u \\
= ABx + Av + u \\
= (AB | u + Av)x.
\]

- $(A | u)(B | v) = (AB | u + Av)$
- The identity transformation is $(I | 0)$. Translation by $v$ is $(I | v)$.
- $(A | u)^{-1} = (A^{-1} | - A^{-1}u)$
Rotations

- Consider \((R | v)\) where \(R = R(\theta) \neq I\) is a rotation.
- Look for a fixed point \(p\), i.e., \((R | v)p = p\)
  \[
  Rp + v = p
  \]
  \[
  \iff p - Rp = v
  \]
  \[
  \iff (I - R)p = v.
  \]
- If \((I - R)\) is invertible, there is a unique solution \(p\).
- \((I - R)\) is invertible because
  \[
  (I - R)x = 0 \iff x = Rx \iff x = 0. \quad \text{(Use the Big Theorem.)}
  \]
- There is a unique fixed point \(p\), and we can write
  \[
  (R | v) = (R | p - Rp).
  \]
- \((R | p - Rp)x = Rx + p - Rp = p + R(x - p)\).  

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  \]
• \((R | p - Rp)x = Rx + p - Rp = p + R(x - p)\).
Picture of Rotation

- \( y = (R \mid p - Rp)x = p + R(x - p) \) says that this isometry is rotation though angle \( \theta \) around the point \( p \), which is called the center of rotation or rotocenter.
Isometries with a Reflection Matrix

- Consider \((S \mid w)\) where \(S = S(\theta)\) is a reflection matrix.
- Let \(u\) and \(v\) be the vectors with \(Su = u\) and \(Sv = -v\).
  We can write \(w = \alpha u + \beta v\), so \((S \mid w) = (S \mid \alpha u + \beta v)\).
- First Case: \(\alpha = 0\). So we have \((S \mid \beta v)\).
- The point \(p = \beta v / 2\) is fixed.
  \[(S \mid \beta v)p = S(\beta v / 2) + \beta v = -\beta v / 2 + \beta v = \beta v / 2 = p.\]
- The fixed points are exactly \(x = p + tu\).
  \[(S \mid 2p)(p + tu) = Sp + tSu + 2p = -p + tu + 2p = p + tu.\]
The Line of Fixed Points

- The line through $p$ parallel to $u$ is parametrized by $x = p + tu$, for $t \in \mathbb{R}$.
Reflection Picture

- We can write a point $x$ as $x = p + tu + sv$. Then $y = (S | 2p)x = p + tu - sv$. 

![Reflection Picture](image)
Reflextions and Glides

- \((S \mid \beta v) = (S \mid 2p)\) is reflection through the mirror line \(M\) given by \(y = p + tu\).
- **Second Case:** \(\alpha \neq 0\). We have \((S \mid \alpha u + \beta v)\).

\[
(S \mid \alpha u + \beta v) = (I \mid \alpha u)(S \mid \beta v)
\]

This is a reflection followed by a translation parallel to the mirror line. This is called a **glide reflection** or just a **glide**. The mirror line of the reflection is called the **glide line**.

- A glide has no fixed points.
A glide with a horizontal glide line, and the translation vector shown in blue.
Classification Theorem

Theorem

Every isometry of the plane falls into one of the following five mutually exclusive classes.

1. The identity.
2. A translation (not the identity).
3. A rotation about some point (not the identity);
4. A reflection through some mirror line.
5. A glide along some glide line.
Exercises on Isometries

Exercises

1. Consider the cases for the product $T_1 T_2$ of two isometries $T_1$ and $T_2$. Describe what happens geometrically (e.g., rotation angle, location of the mirror line, etc.). In what cases do the isometries commute, i.e., when does $T_1 T_2 = T_2 T_1$?

2. Project: For a rotation $(R|v)$ give a geometric description of how to find the rotocenter $p = (I - R)^{-1}v$.

3. As part of the first exercise, given two rotations (with possibly different rotocenters) show that the composition is a rotation or a translation.

4. Project: Given two rotations with different rotocenters give a geometric description of how to find the rotocenter of the composition.