Geometric Transformations and Wallpaper Groups
Vector and Matrix Algebra

Lance Drager

Department of Mathematics and Statistics
Texas Tech University
Lubbock, Texas

2010 Math Camp
Outline

1 Vectors
  Geometric Vectors
  Vectors in Coordinates
  The Dot Product

2 Matrices
  What is a matrix?
  Matrix Multiplication
  Invertible Matrices
  Exercises
  Linear Transformations
  Rotations
  Exercises
Vectors

- Our setting is the Euclidean Plane
- A vector is a quantity that has a magnitude and a direction.
Vector Operations

- We’ll use letters like $u, v, w, \ldots$ to stand for vectors.
- $0$ is the vector with magnitude zero. Just draw it as a point.
- $\|u\|$ denotes the magnitude of the vector $u$.
- A scalar is a quantity with a magnitude but no direction, i.e., just a number.
- Two operations on vectors
  - Scalar Multiplication
  - Vector Addition.
Scalar Multiplication

- Scalar multiplication $sv$.
  - if $s = 0$, then $sv$ is the zero vector.
  - if $s > 0$, then $sv$ has the same direction as $v$ and magnitude $s\|v\|$.
  - if $s < 0$, then $sv$ points in the opposite direction to $v$ and has magnitude $|s| \|v\|$.

- $\|sv\| = |s| \|v\|$. 
Visualizing Scalar Multiplication

- $2v$
- $v$
- $(-1/2)v$
Visualizing Vector Addition

\[ \mathbf{u} + \mathbf{v} \]
Visualizing Vector Subtraction

\[ u - v = u + (-v) \]
Vectors in Coordinates

- We can put a coordinate system on the plane.

\[ y \]

- Distance formula: \( \text{dist}((0, 0), (x, y)) = \sqrt{x^2 + y^2} \)
Components of a Vector

- Coordinates give one-to-one correspondences

Vectors ↔ Points ↔ \((u_1, u_2) \in \mathbb{R}^2\)

- Write a vector as \((u_1, u_2)\). We call \(u_1\) and \(u_2\) the components of \(u\).

\[\|u\| = \sqrt{u_1^2 + u_2^2}\]
Scalar Multiplication in Coordinates

\[ \mathbf{v} = s \mathbf{u} \]

\[ \frac{u_2}{\| \mathbf{u} \|} = \frac{v_2}{\| \mathbf{v} \|} = \frac{v_2}{s \| \mathbf{u} \|} \]

\[ v_2 = s u_2 \]

\[ s(u_1, u_2) = (su_1, su_2) \]
Vector Addition in Coordinates

$$w_1 = u_1 + v_1$$

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$
Standard Basis Vectors

\[(u_1, u_2) = u_1 (1, 0) + u_2 (0, 1) = u_1 e_1 + u_2 e_2\]

\[e_1 = (1, 0), \quad e_2 = (0, 1)\]
A Nonstandard Basis

\[ w = \alpha u + \beta v \]
Rules of Vector Algebra

- \((u + v) + w = u + (v + w)\). (Associative Law)
- \(u + v = v + u\). (Commutative Law)
- \(0 + v = v\). (Additive Identity)
- \(v + (-v) = 0\). (Additive Inverse)
- \(s(u + v) = su + sv\). (Distributive Law)
- \(s(tu) = su + tu\). (Distributive Law)
- \(1u = u\).
Trig before Dot Product
Dot Product

- The angle $\theta$ between two vectors.

- Definition of dot product

$$u \cdot v = \|u\| \|v\| \cos(\theta).$$

$$0 \cdot v = 0.$$

- $u \cdot u = \|u\|^2.$

- $u \cdot v = 0 \iff u \perp v.$
Law of Cosines from Trig

\[ c^2 = a^2 + b^2 - 2ab \cos(\theta). \]
Formula for Dot Product

\[ \|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\theta) \]

\[ = \|u\|^2 + \|v\|^2 - 2u \cdot v \]

\[ 2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2. \]
Dot Product in Components

\[ 2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2 \]

\[ = \|(u_1, u_2)\|^2 + \|(v_1, v_2)\|^2 - \|(u_1 - v_1, u_2 - v_2)\|^2 \]

\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 - [(u_1 - v_1)^2 + (u_2 - v_2)^2] \]

\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 \]

\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1^2 - 2u_1v_1 + v_1^2) - (u_2^2 - 2u_2v_2 + v_2^2) \]

\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2 \]

\[ = 2u_1v_1 + 2u_2v_2. \]

\[ u \cdot v = u_1 v_1 + u_2 v_2. \]
Properties of Dot Product

- Rules of Algebra for Dot Product
  - $u \cdot u = \|u\|^2$.
  - $u \cdot v = v \cdot u$.
  - $(su) \cdot v = s(u \cdot v) = u \cdot (sv)$
  - $u \cdot (v + w) = u \cdot v + u \cdot w$.

- Compute the angle in terms of components
  \[ \theta = \arccos \left( \frac{u \cdot v}{\|u\| \|v\|} \right). \]

- Exercise: If $u$ and $v$ are orthogonal unit vectors (i.e., $\|u\| = 1$) and $w = \alpha u + \beta v$, show $\alpha = w \cdot u$ and $\beta = w \cdot v$. 
Matrices

An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns.

Example:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}, \quad 2 \times 3.$$

If the matrix is called $A$, $a_{ij}$ denotes the entry in row $i$ and column $j$. For example, $a_{23} = 5$. 
• A general matrix

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
    a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{bmatrix}, \quad m \times n.
\]

• Matrix addition and scalar multiplication are defined slotwise, like vectors

\[
\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ 10 & 1 \end{bmatrix} = \begin{bmatrix} 1 + 7 & 3 + 4 \\ 0 + 10 & 5 + 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 10 & 6 \end{bmatrix}
\]
Matrix Multiplication 1

- Size condition for multiplication:

\[(m \times n) \cdot (n \times p) = m \times p.\]

- \((1 \times n) \cdot (n \times 1) = 1 \times 1\)

\[
\begin{bmatrix}
a_1 & a_2 & a_3 & \ldots & a_n
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_n
\end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n.
\]
Matrix Multiplication 2

- Entries of $C = AB$

\[
c_{ij} = \text{Row}_i(A) \text{Col}_j(B).
\]

- Important Cases:

\[
Ax = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
a_{11}x_1 + a_{12}x_2 \\
a_{21}x_1 + a_{22}x_2
\end{bmatrix}.
\]

\[
AB = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix}
= \begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\
a_{21}b_{11} + a_{22}b_{22} & a_{21}b_{12} + a_{22}b_{22}
\end{bmatrix}.
\]
Multiplication is not Commutative!

- If $AB$ and $BA$ are the same size, they must both be $n \times n$ for some $n$.
- Example

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
= \begin{bmatrix}
5 & 7 \\
0 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 2 \\
0 & 5
\end{bmatrix}.
\]
Ways of Looking at Multiplication

- Consider $Ax$

\[
\begin{bmatrix}
 a_{11} & a_{12} \\
 a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2
\end{bmatrix} = 
\begin{bmatrix}
 a_{11}x_1 + a_{12}x_2 \\
 a_{21}x_1 + a_{22}x_2
\end{bmatrix} = 
\begin{bmatrix}
 a_{11} \\
 a_{21}
\end{bmatrix} x_1 +
\begin{bmatrix}
 a_{12} \\
 a_{22}
\end{bmatrix} x_2
= x_1 \text{Col}_1(A) + x_2 \text{Col}_2(A).
\]

- $Ae_1 = \text{Col}_1(A)$ and $Ae_2 = \text{Col}_2(A)$.

- Consider $yB$

\[
\begin{bmatrix}
 y_1 & y_1 \\
 b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
 b_{11} & b_{12} \\
 b_{11} & b_{12}
\end{bmatrix} = 
\begin{bmatrix}
 y_1 b_{11} + y_2 b_{21} & y_1 b_{12} + y_2 b_{22}
\end{bmatrix} = y_1 \text{Row}_1(B) + y_2 \text{Row}_2(B).
\]

- In terms of rows and columns

\[
A[b_1 \mid b_2] = [Ab_1 \mid Ab_2], \quad \begin{bmatrix}
 a_1 \\
 a_2
\end{bmatrix} B = \begin{bmatrix}
 a_1 B \\
 a_2 B
\end{bmatrix}.
\]
Rules of Matrix Algebra

- Rules of Algebra for Matrix Multiplication
  - \((AB)C = A(BC)\). (Associative Law)
  - \(A(B + C) = AB + AC\). (Left Distributive Law)
  - \((A + B)C = AC + BC\). (Right Distributive Law)
  - \((sA)B = s(AB) = A(sB)\).
  - \(A0 = 0\) and \(0B = 0\).
Invertible Matrices 1

- $I_n$ denotes the $n \times n$ identity matrix. It has ones on the diagonal and zeros elsewhere.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- If $A$ is $m \times n$

$$I_mA = A = AI_n.$$

- $A$ is invertible or nonsingular if there is a matrix $B$ such that

$$AB = BA = I.$$
Invertible Matrices 2

- Only square matrices can be invertible.
- $B$ is unique, if it exists, and is denoted $A^{-1}$.
  \[
  A^{-1}A = AA^{-1} = I.
  \]
- $(A^{-1})^{-1} = A$.
- The **Determinant** of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is defined by
  \[
  \det(A) = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.
  \]
Big Theorem on Invertible Matrices

**Theorem**

Let

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

be a matrix. Then, the following conditions are equivalent.

1. A is invertible.
2. The only column vector \( x \) such that \( Ax = 0 \) is \( x = 0 \).
3. \( \det(A) \neq 0 \).

**Proof.**

- (1) \( \implies \) (2). Multiply both sides of \( Ax = 0 \) by \( A^{-1} \).
Big Theorem Proof Continued 1

Proof Continued.

• (2) \implies (3). Instead, do: Not (3) \implies Not (2). Assume \(ad - bc = 0\). Of course, 0\(x\) = 0 for all \(x\). Suppose \(A \neq 0\).

Then at least one of the vectors

\[
\begin{bmatrix}
d \\
-\ b
\end{bmatrix}, \quad \begin{bmatrix}
-\ c \\
\ a
\end{bmatrix}
\]

is not zero. But,

\[
\begin{bmatrix}
a & c \\
\ b & \ d
\end{bmatrix} \begin{bmatrix}
d \\
-\ b
\end{bmatrix} = \begin{bmatrix}
ad - bc \\
bd - bd
\end{bmatrix} = 0,
\]

\[
\begin{bmatrix}
a & c \\
\ b & \ d
\end{bmatrix} \begin{bmatrix}
-\ c \\
\ a
\end{bmatrix} = \begin{bmatrix}
-ac + ac \\
-bc + ad
\end{bmatrix} = 0,
\]

so (2) fails.
Proof Continued.

• (3) \implies (1). Just check that this formula works:

\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.
\]
Exercises on Invertibility

Exercises

Here $A$ and $B$ are $2 \times 2$ matrices.

- If $A$ and $B$ are invertible, so is $AB$ and $(AB)^{-1} = B^{-1}A^{-1}$.
- Show that if $AB = I$, then $B = A^{-1}$ and if $BA = I$ then $B = A^{-1}$.
- $A$ is invertible if and only if the equation $Ax = b$ has a unique solution $x$ for every column vector $b$.
- Show that if $\det(A) = 0$, one of the columns of $A$ is multiple of the other.
- Show by brute force computation (for $2 \times 2$ matrices) that $\det(AB) = \det(A)\det(B)$.
Linear Transformations

Definition
A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is \textbf{linear} if

$$T(u + v) = T(u) + T(v)$$

$$T(su) = sT(u),$$

or, equivalently,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Theorem
If $A$ is a matrix, $T(x) = Ax$ is linear.
The Matrix of a Linear Transformation

Theorem

If $T$ is linear, there is a unique matrix $A$ so that $T(x) = Ax$. In fact, $A = [T(e_1) \mid T(e_2)]$.

Proof.

$$T(x) = T(x_1 e_1 + x_2 e_2)$$
$$= x_1 T(e_1) + x_2 T(e_2)$$
$$= [T(e_1) \mid T(e_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
Rotation is Linear

• Let $T$ be the transformation given by rotation around the origin by a fixed angle $\theta$.
• We can see geometrically that $T$ is linear.
$T(u + v) = T(u) + T(v)$
Matrix of Rotation

- We need to find $T(e_1)$ and $T(e_2)$.
Find $T(e_1)$

- We know $T(e_1)$ from trig:

$$T(e_1) = (\cos(\theta), \sin(\theta))$$
Find $T(e_2)$

- $T(e_2)$ must be $(-\sin(\theta), \cos(\theta))$ or $(\sin(\theta), -\cos(\theta))$. Check the signs.

- $T(e_2) = (-\sin(\theta), \cos(\theta))$. 
The Matrix of a Rotation

**Theorem**

The matrix of rotation through angle $\theta$ is

$$R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}$$

**Theorem**

From the geometry, we have

$$R(\theta)R(\phi) = R(\theta + \phi) = R(\phi)R(\theta)$$
The Addition Laws

Exercises

• From the geometry \([R(\theta)]^{-1} = R(-\theta)\). Compute \([R(\theta)]^{-1}\) from our previous theorem. Compare entries in these matrices to conclude \(\cos(-\theta) = \cos(\theta)\) and \(\sin(-\theta) = -\sin(\theta)\).

• Compare the matrix entries in the equation

\[R(\theta)R(\phi) = R(\theta + \phi).\]

Then, replace \(\phi\) by \(-\phi\) and use the first part of the problem.

You should get the Addition Laws for Sine and Cosine

\[
\begin{align*}
\cos(\theta \pm \phi) &= \cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi), \\
\sin(\theta \pm \phi) &= \sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi).
\end{align*}
\]
• An angle determines a point on the unit circle.
Definition of Sine and Cosine

\[(\cos(\theta), \sin(\theta))\]
The Basic Trig Identity

- A point \((x, y)\) is on the unit circle if and only if
  \[
  \sqrt{x^2 + y^2} = 1 \iff x^2 + y^2 = 1.
  \]

- \((\cos(\theta), \sin(\theta))\) is on the unit circle so
  \[
  [\cos(\theta)]^2 + [\sin(\theta)]^2 = 1
  \]
  \[
  \cos^2(\theta) + \sin^2(\theta) = 1
  \]

- Some people write \(\sin(\theta) = \sin \theta\)
Non-Unit Circle

$$(r \cos(\theta), r \sin(\theta))$$

$$(\cos(\theta), \sin(\theta))$$
Trig

Introduction

Triangles

Let \( (r \cos(\theta), r \sin(\theta)) \) be a point on the unit circle.

- \( \frac{\text{opp}}{\text{hyp}} = \frac{r \sin(\theta)}{r} = \sin(\theta) \)
- \( \frac{\text{adj}}{\text{hyp}} = \frac{r \cos(\theta)}{r} = \cos(\theta) \)