PROBLEM SET

Problems on Matrix Exponentials and Differential equations
Corrected Version!

Math 3351, Fall 2010

Oct. 28, 2010

- Write all of your answers on separate sheets of paper. You can keep the question sheet.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This problem et has 3 problems. There are **0 points total**.

Good luck!

Problem 1.
Consider the system of differential equations

$$\begin{aligned} x_1'(t) &= 15x_1(t) + 16x_2(t) \\ x_2'(t) &= -12x_1(t) - 13x_2(t) \end{aligned} \tag{1}$$

A. Express the system in the matrix form $x' = Ax$.

B. Compute the matrix e^{tA}.

C. Find the solution of the system (1) for arbitrary initial conditions

$$\begin{aligned} x_1(0) &= x_1^0 \\ x_2(0) &= x_2^0. \end{aligned}$$

D. find the solution of the system (1) for the specific initial conditions

$$\begin{aligned} x_1(0) &= 2 \\ x_2(0) &= -5. \end{aligned}$$

Problem 2. In each part, diagonalize the matrix A and compute e^{tA}. (These matrices were diagonalized on a previous set of practice problems!) In the case of complex eigenvalues, present e^{tA} with entries that are obviously real (i.e., not is appear).

A.

$$A = \begin{bmatrix} -16 & 36 & -18 \\ -6 & 14 & -6 \\ 3 & -6 & 5 \end{bmatrix}.$$

B.

$$A = \begin{bmatrix} -2 & -4 & 5 \\ -3 & -3 & 5 \\ -5 & -5 & 8 \end{bmatrix}$$

Problem 3. This problem deals with finding e^{tA} and solving the system $x' = Ax$ in an example where A is not diagonalizable. Let A be given by

$$A = \begin{bmatrix} -2 & 7 & -10 \\ -2 & 6 & -4 \\ -1 & 3 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is $-(\lambda-2)^3$, so there is only one eigenvalue, namely $\lambda = 2$. The eigenspace $E(2)$ is only one dimensional and has a basis given by the vector

$$v_1 = \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, A is certainly not diagonalizable.

To find e^{tA} and solve the system $x' = Ax$, proceed as follows.

A. Solve the system $(A - 2I)x = v_1$. You'll get a one parameter family of solutions. Pick a specific solution and call it v_2 (so v_2 is a constant vector). Thus, $(A - 2I)v_2 = v_1$.

B. Similarly, find a specific vector v_3 so that $(A - 2I)v_3 = v_2$.

C. Let P be the matrix whose columns are v_1, v_2 and v_3. Calculate $J = P^{-1}AP$.

D. Solve the system of differential equations $z' = Jz$ by hand (work from the bottom up) with initial conditions z^0, and thus find e^{tJ}.

E. Calculate $e^{tA} = Pe^{tJ}P^{-1}$. This tells you how to solve the system $x' = Ax$, of course.
