PROBLEM SET

Problems on Matrix Exponentials and Differential equations
Corrected Version!

Math 3351, Fall 2010

Oct. 28, 2010

ANSWERS

Problem 1.

Consider the system of differential equations

$$x_1'(t) = 15x_1(t) + 16x_2(t)$$
$$x_2'(t) = -12x_1(t) - 13x_2(t) \tag{1}$$

A. Express the system in the matrix form $x' = Ax$.

Answer:

We can write the system as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 15 & 16 \\ -12 & -13 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This is in the form $x' = Ax$ with

$$A = \begin{bmatrix} 15 & 16 \\ -12 & -13 \end{bmatrix}.$$

B. Compute the matrix e^{tA}.

Answer:

The matrix A appeared on the second set of practice problems, where we found the characteristic polynomial and found that the eigenvalues of A are -1 and 3. Since there are two distinct eigenvalues of the 2×2 matrix A, A must be diagonalizable.

Let's start with the eigenvalue $\lambda = -1$. We compute that

$$A - (-1)I = A + I = \begin{bmatrix} 16 & 16 \\ -12 & -12 \end{bmatrix}.$$

Using a calculator, we find the RREF of $A + I$ to be

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We need to find the nullspace of R. Call the variables x_1 and x_2. From R, we see that x_1 is a leading variable and x_2 is a free variable, say $x_2 = \alpha$. The first row of R tells us that

$$x_1 + x_2 = 0 \implies x_1 = -x_2 = -\alpha.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We conclude that the eigenspace $E(-1)$ is one dimensional with basis vector

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Next, we do the eigenvalue $\lambda = 3$. We compute

$$A - 3I = \begin{bmatrix} 12 & 16 \\ -12 & -16 \end{bmatrix}$$

The RREF of $A - 3I$ is

$$R = \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}.$$

Again, x_1 is a leading variable and x_2 is free, say $x_2 = \alpha$. The first row of R tells us that

$$x_1 + \frac{4}{3}x_2 = 0 \implies x_1 = -\frac{4}{3}x_2 = -\frac{4}{3}\alpha.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}.$$

Thus, we conclude that the eigenspace $E(3)$ is one dimensional with basis vector

$$u = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}.$$

We can get rid of the fraction here by noting that we can multiply every vector in a basis by a nonzero number to get a new basis. Thus, instead of using u as the basis of $E(3)$, we can use the vector $v_2 = 3u$. Thus, we have

$$v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

as the basis vector for $E(3)$. Thus, we have two eigenvectors v_1 and v_2, which we know form a basis of three dimensional space.

To diagonalize A, we form the matrix P with columns v_1 and v_2, so

$$P = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}.$$

We form the diagonal matrix D, where the diagonal entry in each column of D is the eigenvalue that goes with the corresponding entry of P. Thus,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

You can use your calculator to check $P^{-1}AP = D$.

Next, we want to compute e^{tA}. The rule for computing the exponential of a diagonal matrix gives us

$$e^{tD} = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{3\,t} \end{bmatrix}$$

Since $A = PDP^{-1}$, we have $Pe^{tD}P^{-1} = e^{tPDP^{-1}} = e^{tA}$ Thus,

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} \\ &= \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{-3\,t} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -3\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t} & -4\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t} & 4\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t} \end{bmatrix} \end{aligned}$$

C. Find the solution of the system (1) for arbitrary initial conditions

$$\begin{aligned} x_1(0) &= x_1^0 \\ x_2(0) &= x_2^0. \end{aligned}$$

Answer:

Let

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

Then the solution of the system $x' = Ax$ with initial conditions $x(0) = x^0$ is $x(t) = e^{tA}x^0$. So, in our case,

$$\begin{aligned} \begin{bmatrix} x_1\,(t) \\ x_2\,(t) \end{bmatrix} &= \begin{bmatrix} -3\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t} & -4\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t} & 4\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} \\ &= \begin{bmatrix} \left(-3\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t}\right) x_1^0 + \left(-4\,\mathrm{e}^{-t} + 4\,\mathrm{e}^{3\,t}\right) x_2^0 \\ \left(3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t}\right) x_1^0 + \left(4\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{3\,t}\right) x_2^0 \end{bmatrix} \end{aligned}$$

D. find the solution of the system (1) for the specific initial conditions

$$\begin{aligned} x_1(0) &= 2 \\ x_2(0) &= -5. \end{aligned}$$

Answer:

Substitute $x_1^0 = 2$ and $x_2^0 = -5$ in the last answer. After simplification, we get

$$\begin{bmatrix} x_1\,(t) \\ x_2\,(t) \end{bmatrix} = \begin{bmatrix} 14\,\mathrm{e}^{-t} - 12\,\mathrm{e}^{3\,t} \\ -14\,\mathrm{e}^{-t} + 9\,\mathrm{e}^{3\,t} \end{bmatrix}$$

Problem 2. In each part, diagonalize the matrix A and compute e^{tA}. (These matrices were diagonalized on a previous set of practice problems!) In the case of complex eigenvalues, present e^{tA} with entries that are obviously real (i.e., not is appear).

A.

$$A = \begin{bmatrix} -16 & 36 & -18 \\ -6 & 14 & -6 \\ 3 & -6 & 5 \end{bmatrix}.$$

Answer:

We diagonalized this matrix in the previous problem set, where we found

$$P = \begin{bmatrix} -6 & -1 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We then have

$$e^{tD} = \begin{bmatrix} \mathrm{e}^{-t} & 0 & 0 \\ 0 & \mathrm{e}^{2\,t} & 0 \\ 0 & 0 & \mathrm{e}^{2\,t} \end{bmatrix}$$

and

$$e^{tD} = Pe^{tD}P^{-1} = \begin{bmatrix} 6\,\mathrm{e}^{-t} - 5\,\mathrm{e}^{2\,t} & -12\,\mathrm{e}^{-t} + 12\,\mathrm{e}^{2\,t} & 6\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{2\,t} \\ 2\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{2\,t} & -4\,\mathrm{e}^{-t} + 5\,\mathrm{e}^{2\,t} & 2\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{2\,t} \\ -\mathrm{e}^{-t} + \mathrm{e}^{2\,t} & 2\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{2\,t} & -\mathrm{e}^{-t} + 2\,\mathrm{e}^{2\,t} \end{bmatrix},$$

which I got by using a calculator.

B.

$$A = \begin{bmatrix} -2 & -4 & 5 \\ -3 & -3 & 5 \\ -5 & -5 & 8 \end{bmatrix}$$

Answer:

We diagonalized this matrix on a previous problem set. The eigenvalues are 1, $1+i$ and $1-i$, and we found

$$P = \begin{bmatrix} 3 & 7-i & 7+i \\ 4 & 7-i & 7+i \\ 5 & 10 & 10 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}.$$

We can easily find

$$e^{tD} = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \mathrm{e}^{(1+i)t} & 0 \\ 0 & 0 & \mathrm{e}^{(1-i)t} \end{bmatrix},$$

but to get the final answers to come out in real form, we need to write out the complex exponentials in terms of their real and imaginary parts. Thus, we should write

$$e^{tD} = \begin{bmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \mathrm{e}^{t}\cos(t) + i\mathrm{e}^{t}\sin(t) & 0 \\ 0 & 0 & \mathrm{e}^{t}\cos(t) - i\mathrm{e}^{t}\sin(t) \end{bmatrix}$$

Now, we can compute (by calculator!)

$$e^{tA} == Pe^{tD}P^{-1}$$

$$= \begin{bmatrix} -3\,\mathrm{e}^{t} + 4\,\mathrm{e}^{t}\cos(t) - 3\,\mathrm{e}^{t}\sin(t) & 3\,\mathrm{e}^{t} - 3\,\mathrm{e}^{t}\cos(t) - 4\,\mathrm{e}^{t}\sin(t) & 5\,\mathrm{e}^{t}\sin(t) \\ -4\,\mathrm{e}^{t} + 4\,\mathrm{e}^{t}\cos(t) - 3\,\mathrm{e}^{t}\sin(t) & 4\,\mathrm{e}^{t} - 3\,\mathrm{e}^{t}\cos(t) - 4\,\mathrm{e}^{t}\sin(t) & 5\,\mathrm{e}^{t}\sin(t) \\ -5\,\mathrm{e}^{t} + 5\,\mathrm{e}^{t}\cos(t) - 5\,\mathrm{e}^{t}\sin(t) & 5\,\mathrm{e}^{t} - 5\,\mathrm{e}^{t}\cos(t) - 5\,\mathrm{e}^{t}\sin(t) & \mathrm{e}^{t}\cos(t) + 7\,\mathrm{e}^{t}\sin(t) \end{bmatrix}.$$

Problem 3. This problem deals with finding e^{tA} and solving the system $x' = Ax$ in an example where A is not diagonalizable. Let A be given by

$$A = \begin{bmatrix} -2 & 7 & -10 \\ -2 & 6 & -4 \\ -1 & 3 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is $-(\lambda-2)^3$, so there is only one eigenvalue, namely $\lambda = 2$. The eigenspace $E(2)$ is only one dimensional and has a basis given by the vector

$$v_1 = \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, A is certainly not diagonalizable.

To find e^{tA} and solve the system $x' = Ax$, proceed as follows.

A. Solve the system $(A - 2I)x = v_1$. You'll get a one parameter family of solutions. Pick a specific solution and call it v_2 (so v_2 is a constant vector). Thus, $(A - 2I)v_2 = v_1$.

B. Similarly, find a specific vector v_3 so that $(A - 2I)v_3 = v_2$.

C. Let P be the matrix whose columns are v_1, v_2 and v_3. Calculate $J = P^{-1}AP$.

D. Solve the system of differential equations $z' = Jz$ by hand (work from the bottom up) with initial conditions z^0, and thus find e^{tJ}.

E. Calculate $e^{tA} = Pe^{tJ}P^{-1}$. This tells you how to solve the system $x' = Ax$, of course.

Answer:

I get

$$e^{tA} = \begin{bmatrix} e^{2t} - 4te^{2t} + 6t^2e^{2t} & 7te^{2t} - 15t^2e^{2t} & -10te^{2t} + 6t^2e^{2t} \\ -2te^{2t} + 2t^2e^{2t} & e^{2t} + 4te^{2t} - 5t^2e^{2t} & -4te^{2t} + 2t^2e^{2t} \\ -te^{2t} - t^2e^{2t} & 3te^{2t} + \frac{5}{2}t^2e^{2t} & -t^2e^{2t} + e^{2t} \end{bmatrix}$$
