PROBLEM SET

Problems on Eigenvalues and Diagonalization

Math 3351, Fall 2010

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ANSWERS

Problem 1. In each part, find the characteristic polynomial of the matrix and the eigenvalues of the matrix **by hand computation**.

A.

$$A = \begin{bmatrix} 15 & 16 \\ -12 & -13 \end{bmatrix}$$

Answer:

We first compute $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 15 & 16 \\ -12 & -13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 - \lambda & 16 \\ -12 & -13 - \lambda \end{bmatrix}$$

The characteristic polynomial is the determinant of this matrix, so

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= (15 - \lambda)(-13 - \lambda) + (12)(16) \\ &= -(15)(13) - 15\lambda + 13\lambda + \lambda^2 + (12)(16) \\ &= -195 - 2\lambda + \lambda^2 + 192 \\ &= -3 - 2\lambda + \lambda^2 \end{aligned}$$

Thus, $p(\lambda) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. The eigenvalues of A are the roots of the characteristic polynomial, so the eigenvalues are -1 and 3.

B.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -82 & -11 & -25 \\ 24 & 4 & 6 \end{bmatrix}$$

Answer:

We have

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 0 & 1 \\ -82 & -11 - \lambda & -25 \\ 24 & 4 & 6 - \lambda \end{bmatrix}$$

Expanding along the first row (to take advantage of the 0), we have

$$\begin{aligned}
p(\lambda) &= (2-\lambda)\begin{vmatrix} -11-\lambda & -25 \\ 4 & 6-\lambda \end{vmatrix} + 1 \begin{vmatrix} -82 & -11-\lambda \\ 24 & 4 \end{vmatrix} \\
&= (2-\lambda)[(-11-\lambda)(6-\lambda) - (-25)4] + [-(82)4 - 24(-11-\lambda)] \\
&= (2-\lambda)[-66 + 11\lambda - 6\lambda + \lambda^2 + 100] + [-328 + 264 + 24\lambda] \\
&= (2-\lambda)[\lambda^2 + 5\lambda + 34] + 24\lambda - 64 \\
&= 2\lambda^2 + 10\lambda + 68 - \lambda^3 - 5\lambda^2 - 34\lambda + 24\lambda - 64 \\
&= 4 - 3\lambda^2 - \lambda^3
\end{aligned}$$

Thus, $p(\lambda) = 4 - 3\lambda^2 - \lambda^3$. We want to find the roots of this polynomial. The possible rational roots are the factors of 4, so the possibilities are ± 1, ± 2 and ± 4. It's easy to check that 1 is a root. Long division then gives

$$p(\lambda) = -(\lambda - 1)(\lambda^2 + 4\lambda + 4) = -(\lambda - 1)(\lambda + 2)^2,$$

so the eigenvalues are 1 and -2 (-2 has multiplicity 2 are root of $p(\lambda)$.)

Problem 2. In each part, you are given a matrix A and the eigenvalues of A. Find a basis for each of the eigenspaces. Determine if A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

A. The eigenvalues are -1 and 2 and

$$A = \begin{bmatrix} -16 & 36 & -18 \\ -6 & 14 & -6 \\ 3 & -6 & 5 \end{bmatrix}.$$

Answer:

Use a calculator on this or you'll go insane. Let's start with the eigenvalue $\lambda = -1$. We want to find a basis of the eigenspace $E(-1)$, which is the same thing as the nullspace of the matrix $A - (-1)I = A + I$. We have

$$A + I = \begin{bmatrix} -15 & 36 & -18 \\ -6 & 15 & -6 \\ 3 & -6 & 6 \end{bmatrix}.$$

The Reduced Row Echelon Form of $A + I$ is

$$R = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We find the nullspace of this is the usual way. Let the variables be x_1, x_2, x_3. Then x_1 and x_2 are leading variables, and x_3 is a free variable, say $x_3 = \alpha$. The first row of R tells us that

$$x_1 + 6x_3 = 0 \implies x_1 = -6x_3 \implies x_1 = -6\alpha$$

and the second row tells us that

$$x_2 + 2x_3 = 0 \implies x_2 = -2x_3 = -2\alpha.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6\,\alpha \\ -2\,\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace $E(-1)$ is one dimensional with basis vector

$$\begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}.$$

Next, consider the eigenvalue $\lambda = 2$. We compute that

$$A - 2I = \begin{bmatrix} -18 & 36 & -18 \\ -6 & 12 & -6 \\ 3 & -6 & 3 \end{bmatrix}.$$

The RREF of this is

$$R = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We proceed as before. In this case, x_1 is a leading variable and x_2 and x_3 are free variables. Say $x_2 = \alpha$ and $x_3 = \beta$. The top row of R tells us

$$x_1 - 2x_2 + x_3 = 0 \implies x_1 = 2x_2 - x_3 = 2\alpha - \beta.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2\,\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace $E(2)$ is two dimensional with basis

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We can put the bases of $E(-1)$ and $E(2)$ together to get a basis of three dimensional space, so A is diagonalizable. To diagonalize it, we put the basis we've found into a matrix P and make D the corresponding diagonal matrix with the eigenvalues on the diagonal. So, we can take

$$P = \begin{bmatrix} -6 & -1 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that the diagonal entry in each column of D is the eigenvalue that goes with the corresponding column of P. (We could permute the columns of P, and arrange the eigenvalues in D to match, to get another solution of the problem.)

Check with your calculator that $P^{-1}AP = D$, or equivalently that $A = PDP^{-1}$.

B. The eigenvalues are -1 and 2 and the matrix is

$$A = \begin{bmatrix} 2 & 13 & 29 \\ 0 & 26 & 54 \\ 0 & -12 & -25 \end{bmatrix}.$$

Answer:

First consider the eigenvalue $\lambda = -1$. We calculate that

$$A + I = \begin{bmatrix} 3 & 13 & 29 \\ 0 & 27 & 54 \\ 0 & -12 & -24 \end{bmatrix}.$$

The RREF of this is

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The method for finding the nullspace of R is exactly as above. The result is that $E(-1)$ is one dimensional with basis

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

For eigenvalue $\lambda = 2$, we compute that

$$A - 2I = \begin{bmatrix} 0 & 13 & 29 \\ 0 & 24 & 54 \\ 0 & -12 & -27 \end{bmatrix}$$

The RREF of this is

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let's work through the nullspace computation. There are leading entries in column 2 and column 3, so x_2 and x_3 are leading variables, and x_1 is a free variable, say $x_1 = \alpha$. The second row of R tells us

$$0x_1 + 0x_2 + 1x_3 = 0 \implies x_3 = 0.$$

The first row tells us

$$0x_1 + 1x_2 + 0x_3 = 0 \implies x_2 = 0.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $E(2)$ is one dimensional with basis vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We only have two linearly independent eigenvectors, so we can not find a basis consisting of eigenvectors. Thus, we conclude that A is **not** diagonalizable.

C. The eigenvectors are 1 and $1 \pm i$, and the matrix is

$$A = \begin{bmatrix} -2 & -4 & 5 \\ -3 & -3 & 5 \\ -5 & -5 & 8 \end{bmatrix}$$

Answer:

If you don't believe that some of the eigenvalues are complex, work out the characteristic polynomial on your calculator and use the function `csolve` or `czeros` to find the roots.

For $\lambda = 1$, we can calculate that

$$A - I = \begin{bmatrix} -3 & -4 & 5 \\ -3 & -4 & 5 \\ -5 & -5 & 7 \end{bmatrix}$$

and that the RREF of $A - I$ is

$$R = \begin{bmatrix} 1 & 0 & -3/5 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, x_1 and x_2 are leading variables, and x_3 is a free variable, say $x_3 = \alpha$. Reading up from the bottom of R we have

$$x_2 - \frac{4}{5}x_3 = 0 \implies x_2 = \frac{4}{5}x_3 = \frac{4}{5}\alpha$$
$$x_1 - \frac{3}{5}x_3 = 0 \implies x_2 = \frac{3}{5}\alpha.$$

Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}\alpha \\ \frac{4}{5}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}.$$

We conclude that $E(1)$ is one dimensional with basis vector

$$\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}.$$

We can avoid the mental agony of fractions if we observe the following fact: multiplying each vector in a basis by a nonzero constant yields another basis. In this case, we can multiply our basis vector by 5 to get another basis vector

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

for $E(1)$.

For eigenvalue $\lambda = 1 + i$, we calculate that

$$A - (1+i)I = \begin{bmatrix} -3-i & -4 & 5 \\ -3 & -4-i & 5 \\ -5 & -5 & 7-i \end{bmatrix}.$$

Fortunately, our calculator will handle complex numbers (use the symbol i right above the `catalog` key for $i = \sqrt{-1}$). Thus, we can enter the matrix above in the calculator and find that the RREF is

$$R = \begin{bmatrix} 1 & 0 & -\frac{7}{10} + \frac{1}{10}\, i \\ 0 & 1 & -\frac{7}{10} + \frac{1}{10}\, i \\ 0 & 0 & 0 \end{bmatrix}.$$

It's easy to find the nullspace of this by our usual method. We see that x_1 and x_2 are leading variables and x_3 is free, say $x_3 = \alpha$. The first row of R says

$$x_1 + (-7/10 + i/10)x_3 = 0 \implies x_1 = (7/10 - i/10)\alpha$$

Similarly, the second row says

$$x_2 = (7/10 - i/10)\alpha.$$

Thus, the nullspace of R is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (7/10 - i/10)\alpha \\ (7/10 - i/10)\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 7/10 - i/10 \\ 7/10 - i/10 \\ 1 \end{bmatrix}.$$

We conclude that $E(1+i)$ is one dimensional with basis vector

$$\begin{bmatrix} 7/10 - i/10 \\ 7/10 - i/10 \\ 1 \end{bmatrix}.$$

We can get rid of the fractions by multiplying this by 10, to get another basis vector

$$\begin{bmatrix} 7-i \\ 7-i \\ 10 \end{bmatrix}.$$

for $E(1+i)$.

We're left with the eigenvalue $\lambda = 1 - i$, which is the complex conjugate of $1 + i$. We don't need to do any calculation. Since A is real, conjugation gives a one-to-one correspondence between $E(1+i)$ and $E(1-i)$ that sends bases to bases. Thus, to get a basis of $E(1-i)$, we just take the conjugate of our basis vector for $E(1+i)$. Thus, $E(1-i)$ is one dimensional with basis vector

$$\begin{bmatrix} 7+i \\ 7+i \\ 10 \end{bmatrix}.$$

Since we get a basis of eigenvectors, we conclude that A is diagonalizable and we can take

$$P = \begin{bmatrix} 3 & 7-i & 7+i \\ 4 & 7-i & 7+i \\ 5 & 10 & 10 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}.$$

I invite you to calculate PDP^{-1} to see that it really works.