# PROBLEM SET

Problems on Span, Independence, and Matrix Spaces

Math 3351, Fall 2010

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# ANSWERS

**Problem 1**. Consider the matrix

$$A = \begin{bmatrix} -15 & -80 & -145 & 22 & -181 \\ -89 & -19 & 51 & 50 & 132 \\ 66 & -62 & -190 & 78 & -96 \\ 77 & 81 & 85 & -8 & 150 \end{bmatrix}.$$

A Find a basis of the nullspace of $A$.

B Find a basis of the rowspace of $A$.

C Find a basis of the column space of $A$. Write the other columns of $A$ as linear combinations of these basis vectors.

D What is the rank of $A$?

*Answer*:
The RREF of $A$ is

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find a basis of the nullspace of $R$, which is the same as the nullspace of $A$, solve the system $Rx = 0$. The variables $x_3$ and $x_5$ are free parameters, say $x_3 = \alpha$ and $x_5 = \beta$. The rows of $R$ give us the following equations

$$\begin{aligned} x_4 + 2x_5 = 0 &\implies x_4 = -2x_5 = -2\beta \\ x_2 + 2x_3 + 3x_5 = 0 &\implies x_2 = -2x_3 - 3x_5 = -2\alpha - 3\beta \\ x_1 - x_3 - x_5 = 0 &\implies x_1 = x_3 + x_5 = \alpha + \beta. \end{aligned}$$

Hence, the solutions of the system are given by the two parameter family

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ -2\,\alpha - 3\,\beta \\ \alpha \\ -2\,\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, as basis for the nullspace of $A$ (which is the same as the nullspace of $R$)

is given by the two vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

A basis of the rowspace of $A$ is given by the nonzero rows of the RREF $R$, hence a basis is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Thus, the rowspace has dimension 3.

In the following, let $a_i$ denote the $i$th column of $A$ and let $r_i$ denote the $i$th column of $R$.

The leading entries in $R$ are in columns 1, 2 and 4. Hence the corresponding columns $a_1$, $a_2$ and $a_4$ of $A$ are a basis of the columnspace of $A$. Thus, a basis of the columnspace of $A$ is given by

$$\begin{bmatrix} -15 \\ -89 \\ 66 \\ 77 \end{bmatrix}, \quad \begin{bmatrix} -80 \\ -19 \\ -62 \\ 81 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 50 \\ 78 \\ -8 \end{bmatrix}.$$

Thus, the columnspace has dimension 3.

The columns of $R$ satisfy the same linear relations as the columns of $A$. Looking at the third column of $R$, we see that $r_3 = -r_1 + 2r_2$. Thus, we must have

$$a_3 = -a_1 + 2a_2.$$

Looking at the the fifth column of $R$, we see $r_5 = -r_1 + 3r_2 + 2r_4$, so we also have

$$a_5 = -a_1 + 3a_2 + 2a_4$$

The rank of $A$ is the common dimension of the rowspace and the columnspace of $A$, which is 3 in this case.

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**Problem 2**. Consider the matrix

$$A = \begin{bmatrix} 8 & 9 & 19 & 4 & 22 & -25 \\ 7 & 1 & -4 & 6 & 3 & 10 \\ 5 & 5 & 10 & 2 & 13 & -12 \\ 2 & 7 & 19 & 4 & 12 & -33 \\ 4 & 4 & 8 & 6 & 6 & -14 \\ 4 & 1 & -1 & 6 & 0 & 1 \end{bmatrix}$$

A Find a basis of the nullspace of $A$.

B Find a basis of the rowspace of $A$.

C Find a basis of the columnspace of $A$. Write the other columns of $A$ as linear combinations of these basis vectors.

D What is the rank of $A$?

*Answer*:
The dimension of the nullspace is 3 and the rank is 3.

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**Problem 3**. Consider the vectors

$$v_1 = \begin{bmatrix} 5 \\ 6 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 11 \\ 12 \\ -1 \\ 12 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 12 \\ 14 \\ 3 \\ 4 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 3 \\ 3 \\ 8 \\ 1 \end{bmatrix}.$$

Let $S = \text{span}(v_1, v_2, v_3, v_4, v_5)$, which is a subspace of $\mathbb{R}^4$.

A. Find a basis of $S$. What is the dimension of $S$?

B. Express the vectors in the list $v_1, \ldots, v_5$ that are not part of the basis as linear combinations of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 7 \\ 9 \\ 0 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 12 \\ 13 \\ 27 \\ -1 \end{bmatrix}.$$

Determine if these vectors are in $S$. If the vector is in $S$, express it as a linear combination of the basis vectors found above.

*Answer*:
Put $v_1, \ldots, v_5$ in a matrix and add $w_1$ and $w_2$ at the right. This gives the matrix

$$A = \begin{bmatrix} 5 & 7 & 11 & 12 & 3 & 7 & 12 \\ 6 & 8 & 12 & 14 & 3 & 9 & 13 \\ 2 & 1 & -1 & 3 & 8 & 0 & 27 \\ 0 & 4 & 12 & 4 & 1 & 3 & -1 \end{bmatrix}.$$

The RREF of $A$ is

$$R = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Looking at the first 5 columns, corresponding to the $v_i$'s, we have leading entries in columns 1, 2 and 5. Hence a basis of $S$ is given by $v_1$, $v_2$ and $v_5$.

Use $r_1, \ldots, r_7$ for the columns of $R$. Looking at $R$, we see $r_3 = -2r_1 + 3r_2$. Hence, we must have

$$v_3 = -2v_1 + 3v_2$$

for the expression of $v_3$ in terms of the basis vectors. Similarly we have $r_4 = r_1 + r_2$, so we have

$$v_4 = v_1 + v_2.$$

For $w_1$ and $w_2$, we see that $r_6$ (corresponding to $w_1$) is not in the span of previous columns, so we conclude that $w_1 \notin S$.

Looking at $r_7$ (corresponding to $w_2$), we see that $r_7 = 2r_1 - r_2 + 3r_5$, thus

$$w_2 = 2v_1 - v_2 + 3v_5,$$

so $w_2 \in S$ and this is its expression in terms of the basis vectors.

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**Problem 4**. Consider the vectors

$$v_1 = \begin{bmatrix} 5 \\ 6 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 11 \\ 12 \\ -1 \\ 12 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 12 \\ 14 \\ 3 \\ 4 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 3 \\ 3 \\ 8 \\ 1 \end{bmatrix}.$$

Let $S = \text{span}(v_1, v_2, v_3, v_4, v_5)$, which is a subspace of $\mathbb{R}^5$.

A. Find a basis of $S$. What is the dimension of $S$?

B. Express the vectors in the list $v_1, \ldots, v_5$ that are not part of the basis as linear combinations of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 7 \\ 9 \\ 0 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 12 \\ 13 \\ 27 \\ -1 \end{bmatrix}$$

Determine if these vectors are in $S$. If the vector is in $S$, express it as a linear combination of the basis vectors found above.

*Answer*:
The basis of $S$ is $v_1, v_2, v_3, v_5$. You should find $w_1 \in S$ and $w_2 \notin S$.

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**Problem 5**. Determine if the following vectors are independent or dependent. If they are independent, complete the list to a basis of $\mathbb{R}^5$ by adding on a standard basis vector. If the vectors are dependent, find a linear relation between them.

$$v_1 = \begin{bmatrix} 13 \\ -10 \\ 24 \\ -9 \\ -11 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ -2 \\ 8 \\ 0 \\ -4 \end{bmatrix}$$

*Answer*:
Put the vectors in a matrix and augment it by the identity matrix. This gives

$$V = \begin{bmatrix} 13 & 4 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -10 & -1 & 1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 24 & 1 & 0 & 8 & 0 & 0 & 1 & 0 & 0 \\ -9 & -2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -11 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The RREF of $V$ is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & 0 & 12 & -2 & 23 \\ 0 & 1 & 0 & 0 & 8 & 0 & -23 & 4 & -44 \\ 0 & 0 & 1 & 0 & -10 & 0 & 31 & -9/2 & \frac{119}{2} \\ 0 & 0 & 0 & 1 & 11 & 0 & -33 & 11/2 & -\frac{127}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1/2 & -1/2 \end{bmatrix}.$$

The leading entries of $R$ are in columns 1, 2, 3, 4 and 6. Thus, the corresponding columns of $V$ are a basis for the column space of $V$. It follows that the vectors $v_1$, $v_2$, $v_3$, $v_4$ and $e_2$ are independent. Since there are 5 of them, they form a basis of $\mathbb{R}^5$.

Thus, our original vectors $v_1$,$v_2$,$v_3$ and $v_4$ are independent, and adding the standard basis vector $e_2$ to the list gives a basis of $\mathbb{R}^5$.

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**Problem 6**. Determine if the following vectors are independent or dependent. If they are independent, complete the list to a basis of $\mathbb{R}^5$ by adding on a standard basis vector. If the vectors are dependent, find a linear relation between them.

$$v_1 = \begin{bmatrix} -38 \\ 91 \\ -1 \\ 63 \\ -23 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -63 \\ -26 \\ 30 \\ 10 \\ 22 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 12 \\ 45 \\ -14 \\ 60 \\ -35 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -89 \\ -170 \\ 64 \\ 24 \\ 42 \end{bmatrix}$$

*Answer*:
The vectors are dependent. A linear relation is $-2v_1 + 3v_2 + 2v_3 - v_4 = 0$ (i.e., $v_4 = -2v_1 + 3v_2 + 2v_3$).

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**Problem 7**. Consider the vectors

$$v_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 28 \\ 26 \\ 12 \\ 7 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$$

Do these vectors span $\mathbb{R}^4$? If so, select a basis of $\mathbb{R}^4$ out of this list of vectors.

*Answer*:
The span of these vectors is 3 dimensional, so the vectors do not span all of $\mathbb{R}^4$.

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**Problem 8**. Consider the vectors

$$v_1 = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 8 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 9 \\ 4 \\ 2 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -8 \\ -8 \\ -12 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 0 \\ 7 \\ 2 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 9 \\ 10 \\ 0 \\ 10 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 3 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 42 \\ 22 \\ 47 \\ 8 \end{bmatrix}$$

Do these vectors span $\mathbb{R}^4$? If so, select a basis of $\mathbb{R}^4$ out of this list of vectors.

*Answer*:
These vectors span $\mathbb{R}^4$. The vectors $v_1, v_2, v_4, v_6$ form a basis of $\mathbb{R}^4$.

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