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# EXAM

Exam #3

Math 2360, Second Summer Session 2007

August 3, 2007

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- This is a Take Home Exam. It is due on Wednesday, August 8, by Noon.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g.,  $\sqrt{2}$ , not 1.414).
- You may discuss the problems with other people, but write up the solutions by yourself.
- You will have to use a calculator. In particular, you can use the calculator to do matrix algebra, dot products, and find the RREF of a matrix. Say what you are computing with the calculator and give the result. If there are any questions on when it is legal to use a calculator, ask me.
- This exam has 8 problems. There are **410 points total**.

Good luck!

40 pts.

**Problem 1.** The matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

is invertible. Use row operations to express  $A$  as a product of elementary matrices. You can use a calculator to do the row operations, but you'll have to show each row operation.

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50 pts.

**Problem 2.** Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ -5 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -2 \\ 7 \\ 6 \\ -7 \\ 1 \end{bmatrix}.$$

Let  $S \subset \mathbb{R}^5$  be defined by

$$S = \text{span}(v_1, v_2, v_3, v_4, v_5).$$

A. Find a basis for  $S$ . What is the dimension of  $S$ ?

B. Consider the vectors

$$w_1 = \begin{bmatrix} 9 \\ 6 \\ 4 \\ 1 \\ 8 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \\ 2 \end{bmatrix}.$$

Determine if each of these vectors is in  $S$ . If the vector is in  $S$ , write it as a linear combination of the basis vectors for  $S$  you found in the first part.

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40 pts.

**Problem 3.** In this problem, we're working in the vector space

$$P_3 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \},$$

the space of polynomials of degree less than three. Let  $\mathcal{U}$  be the basis of  $P_3$  given by

$$\mathcal{U} = [x^2 \quad x \quad 1].$$

Let  $T: P_3 \rightarrow P_3$  be the linear transformation defined by

$$T(p(x)) = p'(x) + 2p(x).$$

Find  $[T]_{\mathcal{U}\mathcal{U}}$ , the matrix of  $T$  with respect to the basis  $\mathcal{U}$ .

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80 pts.

**Problem 4.** Let  $\mathcal{U} = [u_1 \ u_2]$  be the basis of  $\mathbb{R}^2$ , where

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A. Find the change of basis matrices  $S_{\mathcal{E}\mathcal{U}}$  and  $S_{\mathcal{U}\mathcal{E}}$ , where  $\mathcal{E}$  is the standard basis of  $\mathbb{R}^2$ .

B. If

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

find  $[v]_{\mathcal{U}}$ , the coordinates of  $v$  with respect to  $\mathcal{U}$ .

C. If

$$[w]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

find  $w$ .

D. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that satisfies

$$\begin{aligned} T(u_1) &= 2u_1 + 3u_2 \\ T(u_2) &= u_1 - u_2. \end{aligned}$$

Find  $[T]_{\mathcal{U}\mathcal{U}}$ , the matrix of  $T$  with respect to  $\mathcal{U}$ , and  $[T]_{\mathcal{E}\mathcal{E}}$ , the matrix of  $T$  with respect to  $\mathcal{E}$ .

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40 pts.

**Problem 5.** Let

$$A = \begin{bmatrix} -4 & 3 \\ -2 & 3 \end{bmatrix}$$

Find the characteristic polynomial and the eigenvalues of  $A$ . (Do not find any eigenvectors.)

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60 pts.

**Problem 6.** In each part, you are given a matrix  $A$  and its eigenvalues. Find a basis for each of the eigenspaces of  $A$  and determine if  $A$  is diagonalizable. If so, find a diagonal matrix  $D$  and an invertible matrix  $P$  so that  $P^{-1}AP = D$ .

A. The matrix is

$$A = \begin{bmatrix} 11 & -3 & -3 \\ 9 & -1 & -3 \\ 27 & -9 & -7 \end{bmatrix}$$

and the eigenvalues are  $-1$  and  $2$ .

B. The matrix is

$$A = \begin{bmatrix} -4 & -1 & 4 \\ -6 & -2 & 7 \\ -6 & -1 & 6 \end{bmatrix}$$

and the eigenvalues are  $-1$  and  $2$ .

C. The matrix is

$$A = \begin{bmatrix} -3 & 2 & 2 \\ -1 & 5 & -4 \\ -2 & 2 & 1 \end{bmatrix}$$

and the eigenvalues are  $-1$ ,  $2 + i$  and  $2 - i$ .

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40 pts.

**Problem 7.** Consider the three vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Apply the Gram-Schmidt Process to these vectors to produce an orthonormal basis of  $\mathbb{R}^3$ .

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60 pts.

**Problem 8.** Let  $S$  be the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

- A. Find an orthonormal basis for  $S$ .
- B. Find an orthonormal basis for  $S^\perp$ .
- C. Determine if each of the following vectors is in  $S$  by computing inner products.

$$w_1 = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \end{bmatrix}.$$

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