70 pts.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 3 & -1 \\ 2 & 1 & 3 & 1 & 5 & -4 \\ 0 & 2 & -2 & 1 & 0 & -3 \\ 1 & 1 & 1 & 0 & 4 & -2 \\ 2 & 0 & 4 & 1 & 4 & -3 \end{bmatrix}$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A.

D. What is the rank of A?

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50 pts.

Problem 2. Let A be a 7×6 matrix and let B be a 5×5 matrix.

A. What is the largest possible value of the rank of A?

B. If the nullspace of A has dimension 2, what is the rank of A?

C. If the columnspace of B has dimension 3, what is the dimension of the nullspace of B?

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40 pts. **Problem 3**. In each part, determine if the given vectors span $\mathbb{R}^3$. If they span $\mathbb{R}^3$, find a sublist which is a basis of $\mathbb{R}^3$

A.

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ -4 \\ -3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}.$$

B.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 6 \\ 2 \end{bmatrix}.$$

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40 pts. **Problem 4**. In each part, you are given a list of four vector in $\mathbb{R}^5$. Determine if the vectors are linearly independent or linearly dependent. If they are linearly dependent, find scalars c_1, c_2, c_3, c_4, not all zero, so that $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.

A.

$$v_1 = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

B.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

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50 pts.

Problem 5. Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -7 \\ -10 \\ 12 \\ -16 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -3 \\ -5 \\ 5 \\ -8 \end{bmatrix}.$$

Let $S \subset \mathbb{R}^4$ be defined by

$$S = \operatorname{span}(v_1, v_2, v_3, v_4, v_5).$$

A. Find a basis for S. What is the dimension of S?

B. Consider the vectors

$$w_1 = \begin{bmatrix} 4 \\ 5 \\ -7 \\ 9 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 5 \\ 2 \\ -8 \\ 9 \end{bmatrix}.$$

Determine if each of these vectors is in S. If the vector is in S, write it as a linear combination of the basis vectors for S you found in the first part.

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60 pts.

Problem 6. In this problem, we're working in the vector space

$$P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\},$$

the space of polynomials of degree less than three. Let $\mathcal{U}$ be the basis of P_3 given by

$$\mathcal{U} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix},$$

and let $\mathcal{V}$ be the basis of P_3 given by

$$\mathcal{V} = \begin{bmatrix} 3x^2 + 2x + 1 & 2x^2 + x + 1 & 2x^2 + 1 \end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$.

B. Let $p(x) = -x^2 + 2x + 5$. Find $[p(x)]_{\mathcal{V}}$, the coordinates of $p(x)$ with respect to $\mathcal{V}$. Write $p(x)$ as a linear combination of the elements of $\mathcal{V}$

C. Suppose that

$$[q(x)]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Find $q(x)$ in the form $ax^2 + bx + c$.

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40 pts.

Problem 7. Let $\mathcal{U} = [u_1 u_2]$ be the basis of $\mathbb{R}2$, where

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis $\mathcal{E}$ is

$$[T]_{\mathcal{E}\mathcal{E}} = A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

B. Find $[T]_{\mathcal{U}\mathcal{U}}$, the matrix of T with respect to the basis $\mathcal{U}$.

EXAM

Exam #2

Math 2360, Summer 2007

July 31, 2007

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **350 points total**.

Good luck!