

---

**EXAM**

Practice Questions for Exam #1

Math 3350, Spring 2006

Feb.12, 2005

---

ANSWERS



60 pts.

**Problem 1.** In each part, find the general solution of the given differential equation.

A.

$$\frac{dy}{dx} = x + xy^2.$$

*Answer:*

We can write the equation as

$$\frac{dy}{dx} = x(1 + y^2),$$

so the variables separate as

$$\frac{dy}{1 + y^2} = x dx.$$

Integrate both sides

$$\int \frac{dy}{1 + y^2} = \int x dx,$$

to get

$$\tan^{-1}(y) = x^2/2 + C,$$

so the solution is

$$y = \tan(x^2/2 + C).$$

B.

$$\frac{dy}{dx} = \frac{x^2 + 2y^2}{xy}$$

*Answer:*

We can rewrite the equation as

$$\frac{dy}{dx} = \frac{x}{y} + 2\frac{y}{x}.$$

Let  $u = y/x$  (where  $y$  is a solution of our equation), then  $y = xu$ , so  $y' = u + xu'$ . We substitute this expression for the left-hand side of the equation, and rewrite the equation as

$$u + x \frac{du}{dx} = \frac{1}{u} + 2u.$$

Subtracting  $u$  from both sides of the equation, we get

$$x \frac{du}{dx} = \frac{1}{u} + u.$$

Putting the right-hand side over a common denominator gives

$$x \frac{du}{dx} = \frac{u^2 + 1}{u}.$$

Separating the variables in this equation, we get

$$\frac{u}{1 + u^2} du = \frac{dx}{x}.$$

Integrating both sides, we have

$$\int \frac{u}{1 + u^2} du = \int \frac{dx}{x}.$$

The integral on the left can be done by the simple substitution  $v = 1 + u^2$ . The result is

$$\frac{1}{2} \ln|1 + u^2| = \ln|x| + C.$$

Multiplying by 2 gives

$$\ln|1 + u^2| = 2\ln|x| + C.$$

Exponentiating both sides gives

$$e^{\ln|1+u^2|} = e^{2\ln|x|+C} = e^C \left( e^{\ln|x|} \right)^2,$$

and so

$$|1 + u^2| = e^C x^2.$$

Eliminating the absolute value signs we get

$$1 + u^2 = \pm e^C x^2.$$

In this equation  $e^C$  can be any positive constant, so  $\pm e^C$  is an arbitrary constant. Thus, we rewrite the equation as

$$1 + u^2 = Cx^2,$$

where  $C$  stands for an arbitrary constant again. Then we have

$$u^2 = Cx^2 - 1.$$

Recalling the definition of  $u$ , we have

$$\frac{y^2}{x^2} = Cx^2 - 1.$$

Multiplying by  $x^2$ , we get

$$y^2 = Cx^4 - x^2.$$

This is an implicit solution to the original equation, so we can stop here. (If you try to take square roots to solve for  $y$ , remember it could be plus or minus the square root of the right-hand side.)

C.

$$(1.1) \quad \frac{dy}{dx} - \tan(x)y = \frac{1}{\cos^3(x)}$$

*Answer:*

This is a first order linear equation. The standard form of a first order linear equation is

$$(1.2) \quad \frac{dy}{dx} + p(x)y = r(x)$$

The integrating factor for (1.2) is

$$F = e^{\int p(x) dx}.$$

Looking at (1.1), we see that  $p(x) = -\tan(x)$ . From calculus,

$$-\int \tan(x) dx = -[-\ln|\cos(x)|] = \ln|\cos|.$$

Thus, the integrating factor is

$$F = e^{\ln|\cos(x)|} = |\cos(x)| \sim \cos(x),$$

dropping the absolute value signs. Multiplying both sides of (1.1) by  $\cos(x)$  (remembering  $\tan(x) = \sin(x)/\cos(x)$ ), we get

$$\cos(x)\frac{dy}{dx} - \sin(x)y = \frac{1}{\cos^2(x)} = \sec^2(x).$$

This equation is the same as

$$\frac{d}{dx}(\cos(x)y) = \sec^2(x).$$

(Check it! This justifies dropping the absolute value signs above.) Integrating both sides, we get

$$\cos(x)y = \int \sec^2(x) dx = \tan(x) + C.$$

Dividing both sides by  $\cos(x)$ , we have

$$y = \sec(x) \tan(x) + C \sec(x),$$

which is the general solution of (1.1).

D.

$$(1.3) \quad \frac{dy}{dx} + y = y^3 e^{3x}$$

*Answer:*

This is a Bernoulli equation. The standard form of a Bernoulli equation is

$$\frac{dy}{dx} + p(x)y = q(x)y^a,$$

and the trick for solving it is to introduce a new dependent variable  $u = y^{1-a}$ .

Comparing with (1.3), we see that in our equation,  $a = 3$ , so we introduce the new dependent variable  $u = y^{1-3} = y^{-2}$ . Then we have

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx} y^{-2} \\ &= -2y^{-3} \frac{dy}{dx} \\ &= -2y^{-3} [y^3 e^{3x} - y] && \text{(from (1.3))} \\ &= -2e^{3x} + 2y^{-2} \\ &= -2e^{3x} + 2u. \end{aligned}$$

Thus,  $u$  satisfies the differential equation

$$\frac{du}{dx} - 2u = -2e^{3x}.$$

This is a first order linear equation, with integrating factor  $e^{-2x}$ . Multiplying both sides of the equation by  $e^{-2x}$ , we get

$$e^{-2x} \frac{du}{dx} - 2e^{-2x} u = -2e^{3x} e^{-2x} = -2e^x.$$

This equation is the same as

$$\frac{d}{dx} \left( e^{-2x} u \right) = -2e^x.$$

Integrating both sides gives

$$e^{-2x} u = -2e^x + C,$$

and so

$$u = -2e^{3x} + Ce^{2x}.$$

Recalling the definition of  $u$ , we have

$$\frac{1}{y^2} = Ce^{2x} - 2e^{3x},$$

and so

$$y^2 = \frac{1}{Ce^{2x} - 2e^{3x}}$$

which is an implicit form of the general solution of (1.3).

E.

$$\frac{dy}{dx} = x(2x + y)^2 - 2.$$

*Answer:*

Introduce a new independent variable  $u = 2x + y$ . Then

$$\frac{du}{dx} = 2 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{du}{dx} - 2.$$

Substituting in the equation, we get

$$\frac{du}{dx} - 2 = xu^2 - 2,$$

so the equation becomes

$$\frac{du}{dx} = xu^2.$$

The variables separate as

$$u^{-2} du = x dx.$$

Integrating both sides gives

$$-\frac{1}{u} = x^2 + C$$

or, equivalently,

$$\frac{1}{u} = C - x^2.$$

Thus, we have

$$u = \frac{1}{C - x^2}.$$

Putting in the value of  $u$ , we have

$$2x + y = \frac{1}{C - x^2},$$

and so, finally,

$$y = \frac{1}{C - x^2} - 2x$$

60 pts.

---

**Problem 2.** A tank contains 100 gallons of water. Initially there are 20 pounds of salt dissolved in the tank. Water flows into the tank at a rate of 10 gallons per minute. Each gallon of incoming water contains 2 pounds of dissolved salt. The mixture in the tank is kept practically uniform by stirring. Water flows out of the tank at a rate of 10 gallons per minute.

What is the limiting value of the amount of salt in the tank as  $t \rightarrow \infty$ ? At what time does the amount of salt in the tank reach 98% of its limiting value? At what time does it reach 99% of the limiting value? (Give approximations of the answers accurate to two decimal places.)

*Answer:*

Let  $y = y(t)$  be the number of pounds of salt dissolved in the tank. We can start the experiment at  $t = 0$ , so  $y(0) = 20$ .

The rate of change of  $y$  is equal to the rate at which salt flows into the tank, minus the rate at which salt flows out of the tank. The rate at which salt flows in is 20 pounds per minute. The concentration of salt in the tank is  $y/100$  pounds per gallon, and so the rate at which salt flows out of the tank is  $10(y/100)$  pounds per minute. Thus, we have

$$\frac{dy}{dt} = 20 - \frac{y}{10},$$

which we can rewrite as

$$\frac{dy}{dt} = \frac{1}{10}(200 - y).$$

Separating the variables in this equation gives

$$\frac{dy}{200 - y} = \frac{1}{10} dt.$$

Integrating this gives

$$-\ln|200 - y| = \frac{1}{10}t + C,$$

which we can rewrite as

$$\ln|200 - y| = -\frac{1}{10}t + C.$$

Exponentiating both sides gives

$$|200 - y| = e^{-t/10+C} = e^C e^{-t/10}.$$

Eliminating the absolute value signs gives

$$200 - y = \pm e^C e^{-t/10}.$$

In this equation,  $\pm e^C$  is an arbitrary constant, call it  $C$  again, so we have

$$(2.1) \quad 200 - y = C e^{-t/10}.$$

Setting  $t = 0$  in this equation gives  $200 - y(0) = C$ . Since  $y(0) = 20$ , we conclude that  $C = 180$ . Putting this value of  $C$  into (2.1) and solving for  $y$  gives

$$y = 200 - 180e^{-t/10}$$

for the amount of salt in the tank at time  $t$ . As  $t$  becomes large,  $e^{-t/10}$  goes to zero, so  $y$  approaches 200 as  $t \rightarrow \infty$ .

To find the time at which  $y$  reaches 98% of 200, we solve the equation

$$0.98(200) = 200 - 180e^{-t/10}$$

for  $t$ . Thus, we want to solve

$$196 = 200 - 180e^{-t/10},$$

which is equivalent to

$$-4 = -180e^{-t/10}$$

or

$$e^{-t/10} = 4/180 = 1/45.$$

Taking logs of both sides gives

$$-t/10 = \ln(1/45),$$

and so

$$t = -10 \ln(1/45) \approx 38.67$$

minutes.

The procedure for finding the time when  $y$  reaches 99% of 200 is the same. The answer is

$$t = -10 \ln(1/90) \approx 45.00$$

minutes.

---

**Problem 3.** An object with a mass of 100 kilograms is dropped from a height of 2000 meters. The force of air resistance on the object is proportional to the velocity, and the terminal velocity of the object is 50 meters per second. How long does it take for the object to reach the ground?

*Answer:*

Measure position  $x$  downward from where the object is released, so the ground is at  $x = 2000$ . The initial velocity of the object is zero. The force in the positive  $x$ -direction is gravity, and air resistance is in the negative  $x$ -direction. Then  $F = ma$  becomes

$$(3.1) \quad m \frac{dv}{dt} = mg - bv,$$

where  $b$  is an unknown constant. Dividing this equation by  $m$ , we get

$$\frac{dv}{dt} = g - kv, \quad \text{where } k = b/m.$$

The variables separate as

$$\frac{dv}{g - kv} = dt,$$

and integrating both sides gives

$$-\frac{1}{k} \ln|g - kv| = t + C,$$

and so

$$\ln|g - kv| = -kt + C$$

Taking the exponential of both sides gives

$$|g - kv| = e^{-kt+C} = e^C e^{-kt}.$$

Getting rid of the absolute values, we get

$$g - kv = \pm e^C e^{-kt}.$$

Since  $C$  is arbitrary,  $\pm e^C$  is arbitrary, call it  $C$  again, so we get

$$g - kv = C e^{-kt}.$$

Using the initial condition that  $v = 0$  at  $t = 0$ , we get  $C = g$ . Thus, we have

$$g - kv = g e^{-kt},$$

or

$$kv = g(1 - e^{-kt})$$

and so

$$v = \frac{g}{k}(1 - e^{-kt}).$$

Using the fact that  $k = b/m$ , this becomes

$$v = \frac{mg}{b}(1 - e^{-kt}).$$

Since  $k$  is positive,  $(1 - e^{-kt}) \rightarrow 0$  as  $t \rightarrow \infty$ , so the terminal velocity of the object is  $mg/b$ . Using the give data, we have

$$\frac{100g}{b} = 50$$

and so  $b = 2g$ . The  $k = b/m = 2g/100 = g/50$ . Thus, finally, the formula for the velocity is

$$v = 50(1 - e^{-gt/50}).$$

Since  $v = dx/dt$ , integrating the last equation and using the initial condition  $x = 0$  at  $t = 0$ , we get

$$x = 50\left(t + \frac{50}{g}e^{-gt/50}\right)$$

The object will hit the ground when  $x = 2000$ , so we have to solve the equation

$$2000 = 50\left(t + \frac{50}{g}e^{-gt/50}\right)$$

for  $t$ , (where  $g = 9.81$ , of course). Using machine assistance, I get  $t = 39.998$  seconds.

Remark: You can also solve equation (3.1) as a first order linear equation.

---

**Problem 4.** Solve the differential equation

$$(x - y + 2) dx + (x + y - 4) dy = 0.$$

*Answer:*

To eliminate the constant terms, make the substitution

$$x = u + h, \quad y = v + k,$$

where  $h$  and  $k$  are constants to be chosen. Note that  $dx/du = 1$ , and so  $dx = du$ . Similarly,  $dy = dv$ .

Substituting in the equation, we get

$$(u + h - v - k + 2) du + (u + h + v + k - 4) dv = 0.$$

The constant term in the first parenthesis is  $h - k + 2$  and the constant term in the second parenthesis is  $h + k - 4$ . Hence, we can eliminate the constant terms by taking  $h$  and  $k$  to be solutions of the system of equations

$$\begin{aligned} h - k + 2 &= 0 \\ h + k - 4 &= 0 \end{aligned}$$

The solution is  $h = 1$  and  $k = 3$ . Thus, we have

$$(4.1) \quad x = u + 1, \quad y = v + 3,$$

and the equation becomes

$$(4.2) \quad (u - v) du + (u + v) dv = 0,$$

with no constant terms.

Solve (4.2) for  $dv/du$ . This gives

$$\frac{dv}{du} = \frac{v - u}{u + v}.$$

Dividing the top and bottom of the right-hand side by  $u$  gives

$$\frac{dv}{du} = \frac{v/u - 1}{1 + v/u},$$

+ so the equation is homogeneous. Let  $z = v/u$ , so  $v = uz$  and  $dv/du = z + u dz/du$ . Then the last equation becomes

$$z + u \frac{dz}{du} = \frac{z - 1}{1 + z},$$

so

$$u \frac{dz}{du} = \frac{z - 1}{1 + z} - z = \frac{z - 1 - z - z^2}{1 + z} = -\frac{1 + z^2}{1 + z}.$$

The variables separate as

$$(4.3) \quad \frac{1 + z}{1 + z^2} dz = -\frac{du}{u}.$$

The integral on the left is easy,

$$\int \frac{1 + z}{1 + z^2} dz = \int \frac{z}{1 + z^2} dz + \int \frac{1}{1 + z^2} dz = \frac{1}{2} \ln(1 + z^2) + \tan^{-1}(z) + C$$

Thus, integrating both sides of (4.3), we get

$$\frac{1}{2} \ln(1 + z^2) + \tan^{-1}(z) = -\ln|u| + C$$

Multiply both sides by 2, to get

$$\ln(1 + z^2) + 2 \tan^{-1}(z) = -2 \ln|u| + C.$$

Substituting  $z = v/u$ , we get

$$\ln(1 + v^2/u^2) + 2 \tan^{-1}(v/u) = -2 \ln|u| + C.$$

We can simplify this in the following steps.

$$\begin{aligned} \ln\left(\frac{u^2 + v^2}{u^2}\right) + 2 \tan^{-1}(v/u) &= -2 \ln|u| + C \\ \ln(u^2 + v^2) - \ln(u^2) + 2 \tan^{-1}(v/u) &= -2 \ln|u| + C \\ \ln(u^2 + v^2) - 2 \ln|u| + 2 \tan^{-1}(v/u) &= -2 \ln|u| + C \\ \ln(u^2 + v^2) + \tan^{-1}(v/u) &= C. \end{aligned}$$

From (4.1), we have

$$u = x - 1, \quad v = y - 3.$$

Substituting these into our solution, we get the final answer

$$\ln((x - 1)^2 + (y - 3)^2) + 2 \tan^{-1}\left(\frac{y - 3}{x - 1}\right) = C.$$

60 pts.

---

**Problem 5.** Consider the differential equation

$$(*) \quad y \, dx + 3x \, dy = 0.$$

1. Find an integrating factor for  $(*)$  that is a function of  $y$  alone and use it to solve the equation.

*Answer:*

The equation is of the form  $P(x, y) \, dx + Q(x, y) \, dy = 0$ , where  $P = y$  and  $Q = 3x$ .

To check for an integrating factor that is a function of  $y$  alone, we use Theorem 3 on page 70 of the book. So, we calculate

$$\frac{1}{P} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \frac{1}{y} (3 - 1) = \frac{2}{y}.$$

This is a function of  $y$  alone, call it  $S(y)$ . According to the theorem, an integrating factor is given by

$$\begin{aligned} F(y) &= e^{\int S(y) \, dy} \\ &= e^{\int (2/y) \, dy} \\ &= e^{2 \ln |y|} \\ &= |y|^2 \\ &= y^2. \end{aligned}$$

Multiplying both sides of  $(*)$  by  $y^2$  gives

$$(5.1) \quad y^3 \, dx + 3xy^2 \, dy = 0.$$

We have

$$\frac{\partial}{\partial y}(y^3) = 3y^2 = \frac{\partial}{\partial x}(3xy^2),$$

so (5.1) is exact. To solve it, we want to find a function  $u = u(x, y)$  so that

$$(5.2) \quad \frac{\partial u}{\partial x} = y^3$$

and

$$(5.3) \quad \frac{\partial u}{\partial y} = 3xy^2.$$

Integrating (5.2) with respect to  $x$ , we conclude that

$$(5.4) \quad u = xy^3 + f(y),$$

for some function  $f(y)$  that depends only on  $y$ . Differentiating the last equation with respect to  $y$  gives

$$(5.5) \quad \frac{\partial u}{\partial y} = 3xy^2 + f'(y).$$

Comparing (5.5) with (5.3), we see that  $f'(y) = 0$ . Thus,  $f(y)$  is a constant, which we can take to be zero. Setting  $f(y) = 0$  in (5.4), we get  $u = xy^3$ . The solutions of (\*) are the level curves of  $u$ , so the general solution of (\*) is

$$(5.6) \quad xy^3 = C,$$

where  $C$  is an arbitrary constant.

2. Find an integrating factor for (\*) that is a function of  $x$  alone and use it to solve the equation.

*Answer:*

Use Theorem 3 on page 70 of the book. We calculate

$$\frac{1}{Q} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{1}{3x} (1 - 3) = -\frac{2}{3} \frac{1}{x}.$$

This is a function of  $x$  alone, call it  $R(x)$ . According to the theorem, an integrating factor is

$$\begin{aligned} F &= e^{\int R(x) dx} \\ &= e^{-\frac{2}{3} \int dx/x} \\ &= e^{(-2/3) \ln(x)} \\ &= |x|^{-2/3} \\ &\sim x^{-2/3}. \end{aligned}$$

Multiplying both sides of (\*) by  $x^{-2/3}$  gives the equation

$$(5.7) \quad x^{-2/3}y dx + 3x^{1/3} dy = 0.$$

Since

$$\frac{\partial}{\partial y}(x^{-2/3}y) = x^{-2/3} = \frac{\partial}{\partial x}(3x^{1/3}),$$

equation (5.7) is exact.

Thus we want to find a function  $u$  so that

$$(5.8) \quad \frac{\partial u}{\partial x} = x^{-2/3}y$$

and

$$(5.9) \quad \frac{\partial u}{\partial y} = 3x^{1/3}.$$

Integrating (5.9) with respect to  $y$  gives  $u = 3x^{1/3}y + f(x)$ . Comparison with (5.8) shows that we can take  $f(x)$  to be zero. Thus, the general solution of (\*) is

$$(5.10) \quad 3x^{1/3}y = C,$$

where  $C$  is an arbitrary constant.

3. Are your answers for the first two parts of the problem compatible? Explain.

*Answer:*

Since we're solving the same equation, (5.6) and (5.10) should represent the same family of curves. It's easy to see that this is the case. In (5.10), divide both sides of the equation by 3 to get

$$x^{1/3}y = C/3.$$

Since  $C/3$  is still arbitrary, we can rewrite this as

$$x^{1/3}y = C$$

where  $C$  is still an arbitrary constant. Cubing both sides of the equation gives

$$xy^3 = C^3$$

In this equation,  $C^3$  can still be anything, so this equation is the same as (5.6).

I should remark that sometimes the two solutions one gets by this method are not strictly equivalent. It's possible that one gives a smaller family of curves, because of domain restrictions. For example, the curves defined by the equation

$$(5.11) \quad x\sqrt{y} = C$$

are all members of the family of curves defined by

$$(5.12) \quad x^2y = C.$$

But not every curve in (5.12) satisfies (5.11) because (5.11) only makes sense when  $y$  is positive.