EXAM

Exam #3

Math 2360, Spring 2006

April 21, 2006

- This is a Take Home Exam. It is due on Tuesday, May 2, by 5 p.m.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- You may discuss the problems with other people, but write up the solutions by yourself.
- You will have to use a calculator. In particular, you can use the calculator to do matrix algebra, dot products, find the RREF of a matrix, and do integrals. Say what you are computing with the calculator and give the result. If there are any questions on when it is legal to use a calculator, ask me.
- This exam has 7 problems. There are **340 points total**.

Good luck!

60 pts.

Problem 1. Consider the vectors

$$u_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^2$.

A. Show that $\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ is an ordered basis of $\mathbb{R}^2$. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$.

B. Let $v \in \mathbb{R}^2$ be the vector

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find $[v]_{\mathcal{U}}$, the coordinate vector of v with respect to the basis $\mathcal{U}$.

C. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis is

$$[T]_{\mathcal{E}\mathcal{E}} = \begin{bmatrix} 18 & -30 \\ 10 & -17 \end{bmatrix}$$

Find $[T]_{\mathcal{U}\mathcal{U}}$, the matrix of T with respect to the basis $\mathcal{U}$.

D. Find $[T(v)]_{\mathcal{U}}$, the coordinates of $T(v)$ with respect to $\mathcal{U}$, where v is the vector in part B.

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60 pts.

Problem 2. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

and the eigenvalues are 2 and 3.

B. The matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ -2 & -1 & 5 \end{bmatrix}$$

and the eigenvalues are 2 and 3.

C. The matrix is

$$A = \begin{bmatrix} -3 & 5 & -5 \\ -8 & 9 & -6 \\ -2 & 3 & -2 \end{bmatrix}$$

and the eigenvalues are 2, $1 + 2i$ and $1 - 2i$.

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40 pts.

Problem 3. Consider the three vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Apply the Gram-Schmidt Process to these vectors to produce an orthonormal basis of $\mathbb{R}^3$.

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40 pts.

Problem 4. In this problem, we work in the space $C[0,1]$ of continuous functions on the interval $[0,1]$. We define the inner product on this space by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)x^2\,dx.$$

(This is called a weighted inner product with x^2 as the weight function.) Consider the functions

$$v_1 = 1, \qquad v_2 = x, \qquad v_3 = x^2,$$

and let S be the subspace of $C[0,1]$ spanned by v_1, v_2 and v_3. In this problem, you can use a calculator to perform the integrals.

A. Apply the Gram-Schmidt Process to v_1, v_2 and v_3 get an orthonormal basis of S.

B. Let

$$f(x) = \sqrt{x}.$$

Find the function g in S that is closest to f. What is $\|f - g\|$?

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60 pts.

Problem 5. Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

A. Find an orthonormal basis for S.

B. Find an orthonormal basis for $S^\perp$.

C. Determine if each of the following vectors is in S by computing inner products.

$$w_1 = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 4 \end{bmatrix}.$$

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40 pts. **Problem 6**. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 1 & 0 & -1 \\ 2 & 1 & 1 & -4 \\ 2 & 2 & 1 & -3 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$

Show that the system $Ax = b$ is inconsistent. Find the vectors x that solve the system $Ax = b$ in the least squares sense. Find the residual $\|Ax - b\|$, where x is a least squares solution.

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40 pts. **Problem 7**. Find the line $y = mx + b$ that best fits the following list of data points in the least squares sense,

$$\begin{aligned} (x_1, y_1) &= (1, 3.1) \\ (x_2, y_2) &= (1.5, 3.94) \\ (x_3, y_3) &= (2, 4.8) \\ (x_4, y_4) &= (2.5, 6.23) \\ (x_5, y_5) &= (3, 6.91). \end{aligned}$$

Give your answers for m and b in decimal form to 3 decimal places.
