EXAM

Take-home Exam

Math 5399, Second Summer 2005

July 22, 2005

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 6 problems. There are **0 points total**.

Good luck!

Problem 1. Do Problem 6.1-6. Possible hint: What do the powers A, A^2, ... tell you about the graph of A?

Problem 2. Find all $n \times n$ matrices C so that $CA = AC$ for all $n \times n$ matrices A.

Problem 3. Recall that a matrix A over $\mathbb{K}$ is normal if $A^*A = AA^*$. If A is normal, we have the spectal decompostion

$$A = \sum_{j=1}^{k} \lambda_j P_j,$$

where $\lambda_1 \ldots \lambda_k$ are the distinct eigenvalues of A, as described in the book.

A. If A is diagonalizable $n \times n$ matrix over $\mathbb{K}$, there is some inner product on $\mathbb{K}^n$ that makes A normal.

B. If A is normal (as above) show that

$$A^p = \sum_{j=1}^{k} \lambda_j^p P_j, \qquad p = 1, 2, 3, \ldots .$$

C. If A is normal and $p(z) \in \mathbb{C}[z]$ is a polynomial, then

$$p(A) = \sum_{j=1}^{k} p(\lambda_j) P_j.$$

D. Let A be normal. If f is a function $f\colon \operatorname{spec}(A) \to \mathbb{C}$, define $f(A)$ by

$$f(A) = \sum_{j=1}^{k} f(\lambda_j) P_j.$$

   i. Show that $f \mapsto f(A)$ as an algebra homomorphism from the algebra of complex-valued functions on $\operatorname{spec}(A)$ to the algebra of $n \times n$ matrices over $\mathbb{C}$.

   ii. For any fuction f on $\operatorname{spec}(A)$, there is a polynomial $p(z)$ so that $f(A) = p(A)$.

E. Let A be a normal $n \times n$ matrix and let $C(A)$ be the collection of all $n \times n$ complex matrices B so that $AB = BA$.

   i. Describe $C(A)$.

ii. Let $C^2(A)$ be the collection of all matrices C so that $BC = CB$ for all $B \in C(A)$. Describe $C^2(A)$.

Problem 4. Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. One was of looking at the Jordan Canonical Form is the following:

Theorem 1. *There are linear transformations S and N so that*

1. $T = S + N$.
2. $SN = NS$.
3. *S is diagonalizable and N is nilpotent.*

A If N_1 and N_2 are nilpotent and $N_1 N_2 = N_2 N_1$ then $N_1 + N_2$ is nilpotent.

B The transformations S and N in Theorem 1 are unique.

C If a linear transformation $L\colon \mathbb{C}^n \to \mathbb{C}^n$ is invertible, $L^{-1} = p(L)$ for some polynomial $p(z)$.

D The linear transformations S and N in Theorem 1 are polynomials in T (you don't have to write out an explicti polynomial).

E If T is a linear transformation and $p(z)$ is a polynomial then $\mathrm{spec}(p(T)) = p(\mathrm{spec}(T))$.

F If T is a linear transformation and

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

is a complex powerseries such that $\mathrm{spec}(T)$ lies (strictly) inside the circle of convergence of the series, then

$$f(T) = \sum_{k=0}^{\infty} c_k A^k$$

converges and $\mathrm{spec}(f(T)) = f(\mathrm{spec}(T))$.

Problem 5.

Let A be an $m \times n$ matrix over $\mathbb{C}$. Suppose that B is an $n \times m$ matrix that satisfies the Moore-Penrose conditions

1. $ABA = A$.

2. $BAB = B$.

3. $(AB)^* = AB$.

4. $(BA)^* = BA$.

Show that $B = A^+$, where A^+ is the generalized inverse defined in the book.

Problem 6. Find the least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find A^+ for the 3×2 matrix on the left-hand side.
