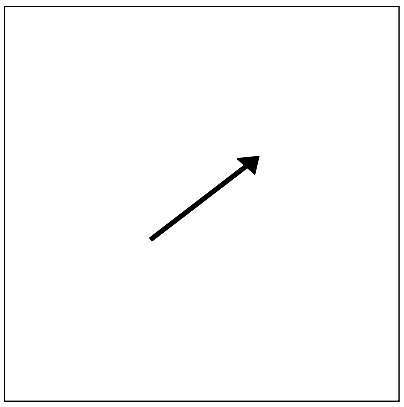
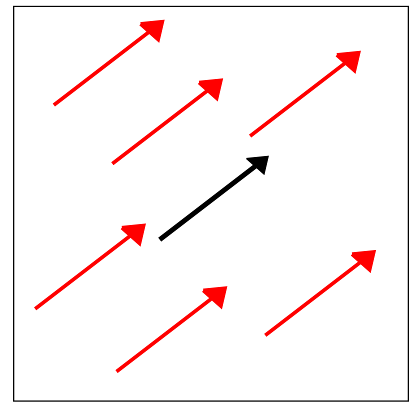
Vectors

- A *vector* in the plane is a quantity that has both a magnitude and a direction. We can represent it by an arrow.



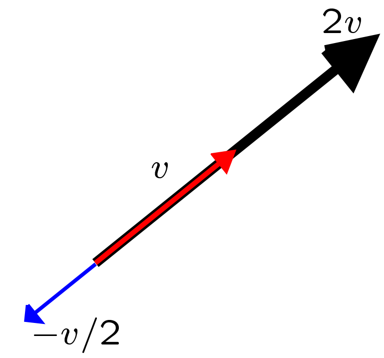
- Arrows that have the same direction and length represent the same vector.



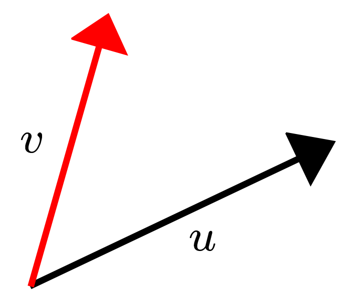
- A special case is the zero vector which has zero magnitude and no particular direction (we just draw it as a dot),
- We'll usually denote a vector by a lower case letter. If u is a vector, $\|u\|$ denotes the magnitude of u.
- A *scalar* is a quantity that has magnitude but no direction, i.e., just a

number.

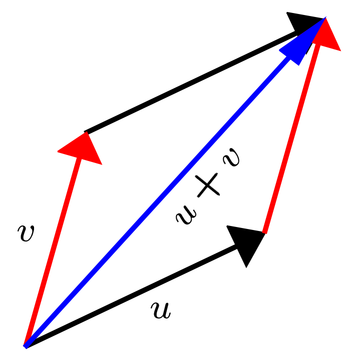
- If v is a vector, $-v$ denotes the vector that has the same length but the opposite direction. Thus, $\|-v\| = \|v\|$.
- If v is a vector an s is a scalar we define the vector sv as follows.
  - If $s = 0$, sv is the zero vector.
  - If $s > 0$, sv has the same direction as v and $\|sv\| = s\|v\|$.
  - If $s < 0$, then $sv = |s|(-v)$, i.e., it points in the direction opposite to v and $\|sv\| = |s|\,\|v\|$.



- If u and v are vectors, as in the picture,

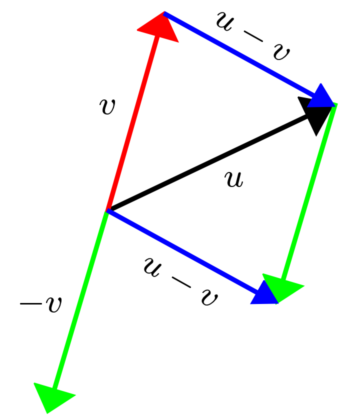


we define $u + v$ by the *parallelogram law*, as in the following diagram.

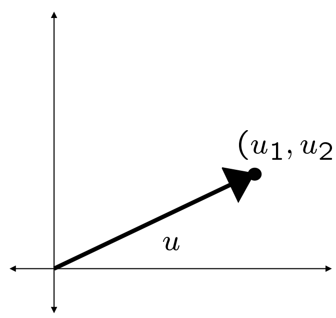


The diagram shows that vector addition is commutative, i.e., $u+v=v+u$.

- Subtraction $u-v$ is defined as $u+(-v)$, but the diagram is worth drawing

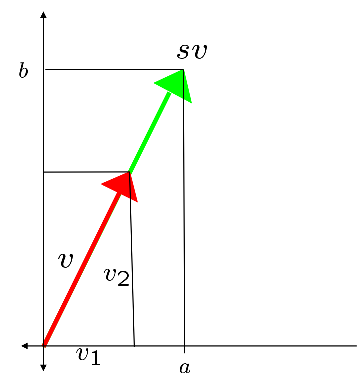


- A choice of a coordinate system sets up a 1-1 correspondence between points in the plane and pairs of real numbers. It also gives a 1-1 correspondence between vectors and pairs of real numbers.



- We just write $u = (u_1, u_2)$.
- The magnitude of the vector is given by $\|u\| = \sqrt{u_1^2 + u_2^2}$.
- The set of ordered pairs of real numbers is denoted $\mathbb{R}^2$.

- How do scalar multiplication and vector addition look in coordinates? Consider the case shown in the diagram



so $s > 0$. Let the coordinates of the tip of sv be (a, b). From similar triangles we have

$$\frac{v_1}{\|v\|} = \frac{a}{\|sv\|} = \frac{a}{s\|v\|}.$$

from which we conclude that

$$a = sv_1.$$

Similarly, we have

$$\frac{v_2}{\|v\|} = \frac{b}{s\|v\|}$$

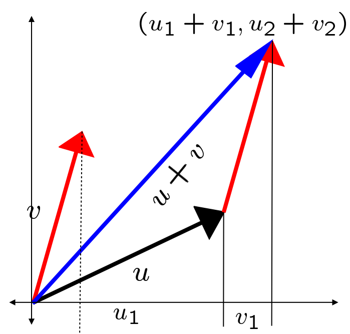
and so

$$b = sv_2.$$

Thus, the components of sv are (sv_1, sv_2). In other words,

$$s(v_1, v_2) = (sv_1, sv_2).$$

For vector addition, consider the diagram

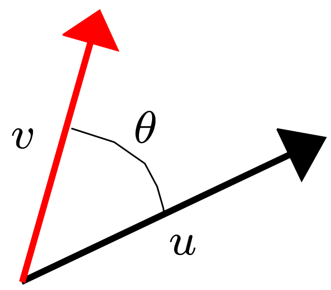


We conclude

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2).$$

- The vector operations satisfy the following rules
  1. $(u + v) + w = u + (v + w)$ (Associative Law)
  2. $u + v = v = v + u$. (Commutative Law)
  3. $0 + v = v$. (Additive identity)
  4. $v + (-v) = 0$. (Additive inverse)
  5. $s(tv) = (st)v$. (Associative Law)
  6. $(s + t)v = sv + tv$. (Distributive Law)
  7. $s(u + v) = su + sv$. (Distributive Law)
  8. $1v = v$.

- We can define the angle between two vectors.



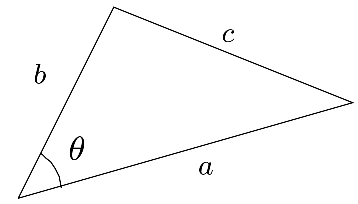
- The *dot product* of two vectors is defined by

$$u \cdot v = \|u\| \, \|v\| \cos(\theta).$$

(This is zero if u or v is zero.) Observe

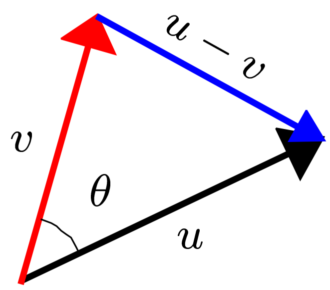
$$u \cdot u = \|u\|^2.$$

Recall the law of cosines:



$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Apply this to the diagram



We get

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \, \|v\| \cos(\theta),$$

which we recognize as

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2u \cdot v.$$

Thus, we get

$$
\begin{aligned}
2u \cdot v &= \|u\|^2 + \|v\|^2 - \|u - v\|^2 \\
&= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\
&\quad - [(u_1 - v_1)^2 + (u_2 - v_2)^2] \\
&= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\
&\quad - (u_1^2 - 2u_1v_1 - v_1^2) - (u_2^2 - 2u_2v_2 + v_2^2) \\
&= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\
&\quad - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2 \\
&= 2u_1v_2 + 2u_2v_2.
\end{aligned}
$$

Thus, we have the formula

$$u \cdot v = u_1v_1 + u_2v_2.$$

- Properties of the dot product.
  1. $u \cdot v = v \cdot u$.
  2. $(su) \cdot v = s(u \cdot v)$.
  3. $(u + v) \cdot w = u \cdot w + v \cdot w$.

- We can use this to compute angles between vectors in terms of the components

$$\cos(\theta) = \frac{u \cdot v}{\|u\| \, \|v\|}.$$

Matrices

- An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

  In this notation, a_{ij} denotes the entry in row i and column j.

- To multiply a matrix by a scalar, multiply each entry by the scalar

- Matrices of the same size can be added by adding corresponding entries, like vectors.

- An $m \times n$ matrix A and an $n \times p$ matrix B can be multiplied to give an $m \times p$ matrix AB.

  For the product of an $1 \times n$ matrix (row vector) and a $n \times 1$ matrix (column vector) the product should be 1×1, i.e., just a number. The definition is

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} =$$
$$a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n.$$

  If A is $m \times n$ and B is $n \times p$, $C = AB$ is $m \times p$ and is given by

$$c_{ij} = \mathrm{Row}_i(A)\,\mathrm{Col}_j(B).$$

- Here are the two cases that will be most important to us. First a 2×2 matrix A times a two entry column vector u

$$Au = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix},$$

yielding a two entry column vector. Second, the product of two 2×2 matrices A and B

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

- Matrix multiplication is *not* commutative!

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$

- **Exercise:** Let A and B be 2×2 matrices. Write $B = \left[\begin{array}{c|c} b_1 & b_2 \end{array}\right]$ where b_1 and b_2 are the columns of B considered as column vectors. Show $AB = \left[\begin{array}{c|c} Ab_1 & Ab_2 \end{array}\right]$. Similarly, write

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where a_1 and a_2 are the rows of A considered as row vectors. Show that

$$AB = \begin{bmatrix} a_1 B \\ a_2 B \end{bmatrix}.$$

- Rules of Matrix Algebra. Matrix addition and scalar multiplication obey the same rules as addition and scalar multiplication of vectors. In addition, we have the following rules (The matrices are assumed to have compatible sizes)
  1. $A(BC) = (AB)C$. (Multiplication is Associative)
  2. $A(B + C) = AB + AC$. (Left Distributive Law)
  3. $(A + B)C = AC + BC$. (Right Distributive Law).
  4. $(sA)B = s(AB) = A(sB)$
- The 2×2 identity matrix is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

  If A and B have compatible sizes, $IA = A$ and $BI = B$.

- A (2×2) matrix A is *invertible* if there is a (2×2) matrix B so that

$$AB = BA = I.$$

  It's easy to show there is at most one matrix B with this property. If B exists, we write $B = A^{-1}$. Note that if A^{-1} exists, $(A^{-1})^{-1} = A$.

- The *determinant* of the 2×2 matrix

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

  is defined by

$$\det(A) = ad - bc.$$

- **Theorem** Let A be a 2×2 matrix. Then, the following conditions are equivalent.
  1. A is invertible.
  2. The only vector v such that $Av = 0$ is $v = 0$.
  3. $\det(A) \neq 0$.

  To show that $(1) \implies (2)$, suppose (1) holds and that $Av = 0$. Then we have $A^{-1}Av = A^{-1}0 = 0$. Then $0 = A^{-1}Av = Iv = v$, so $v = 0$. Thus, (2) holds.

  To show that $(2) \implies (3)$, it will suffice to show that $\neg(3) \implies \neg(2)$. So assume $\det(A) = ad - bc = 0$. If $A = 0$ then $Av = 0$ for all v so (2) fails. So, suppose $A \neq 0$.

  Notice that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} d \\ -b \end{bmatrix} = \begin{bmatrix} ad - bc \\ bd - bd \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} -c \\ a \end{bmatrix} = \begin{bmatrix} -ac + ac \\ -bc + ad \end{bmatrix} = 0.$$

  If $A \neq 0$, one of these vectors is nonzero, so (2) fails.

  Finally, to show that $(3) \implies (1)$, assume that $\det(A) \neq 0$. Just check that the following formula works

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

- **Exercise:** Define the *trace* of A, denoted $\operatorname{tr}(A)$, by

$$\operatorname{tr}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = a + d,$$

  (i.e., the sum of the diagonal entries).

1. Show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

(Use brute force.)

2. Use the first part to show

$$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A).$$

- **Exercise:** $\det(A) = 0$ if and only if one of the columns of A is a multiple of the other.

Linear Transformations

- Let e_1 and e_2 be the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_1 e_2 + u_2 e_2.$$

- If A is a 2×2 matrix then $Ae_1 = \mathrm{Col}_1(A)$ and $Ae_2 = \mathrm{Col}_2(A)$.

- A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is *linear* if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all vectors u and v and scalars α and β.

- An example is the transformation $T(v) = Av$ for a 2×2 matrix A.

- **Theorem** If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear, there is a unique matrix A such that $T(v) = Av$. In fact,

$$A = \left[\ T(e_1) \ \middle|\ T(e_2)\ \right].$$

To see this, suppose T is linear and v is a vector. Then,

$$\begin{aligned} T(v) &= T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= T(v_1 e_1 + v_2 e_2) \\ &= v_1 T(e_1) + v_2 T(e_2) \\ &= \left[\ T(e_1)\ \middle|\ T(e_2)\ \right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \end{aligned}$$