# EXAM

Practice Questions for the Final Exam

Math 3350, Spring 2004

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# ANSWERS

These are some practice problems from Chapter 10, Sections 1–4. See previous practice problem sets for the material before Chapter 10.

**Problem 1**. Let $f(x)$ be the function of period $2L = 4$ which is given on the interval $(-2, 2)$ by

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2 - x, & 0 < x < 2. \end{cases}$$

Find the Fourier Series of $f(x)$.

*Answer*:
The function is neither even nor odd. The Fourier coefficients are calculated as follows.

For $a_0$, we have

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &= \frac{1}{4}\int_{-2}^{2} f(x)\,dx \\ &= \frac{1}{4}\int_{0}^{2} (2-x)\,dx, \qquad \text{since } f(x){=}0 \text{ on } (-2,0) \\ &= \frac{1}{2}. \end{aligned}$$

For $a_n$ with $n \geq 1$, we have

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{1}{2}\int_{0}^{2} (2-x)\cos\left(\frac{n\pi}{2}x\right)dx \\ &= -\frac{2}{n^2\pi^2}[1-(-1)^n] \\ &= \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n^2\pi^2}, & n \text{ odd.} \end{cases} \end{aligned}$$

Finally, for $b_n$ we get

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{1}{2}\int_{0}^{2} (2-x)\sin\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{2}{n\pi}. \end{aligned}$$

For the cosine terms in the series, we use $2k+1$ to run over the odd integers. Thus, the Fourier Series of $f(x)$ is

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left(\frac{(2k+1)\pi}{2}x\right) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin\left(\frac{k\pi}{2}x\right).$$

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**Problem 2**. Let $f(x)$ be the function of period $2L = 2$ which is given on the interval $(-1,1)$ by $f(x) = 1 - x^2$.

Find the Fourier Series of $f(x)$.

*Answer*:
The function is even.

Using the fact the function is even, we get

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &= \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx \\ &= \int_{0}^{1}(1-x^2)\,dx \\ &= \frac{2}{3}. \end{aligned}$$

Again using the fact that the function is even, we get

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= \int_{-1}^{1}(1-x^2)\cos(n\pi x)\,dx \\ &= 2\int_{0}^{1}(1-x^2)\cos(n\pi x)\,dx \\ &= \frac{4(-1)^{n+1}}{n^2\pi^2}. \end{aligned}$$

For the $b_n$'s, we have

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \int_{-1}^{1}(1-x^2)\sin(n\pi x)\,dx \\ &= 0, \end{aligned}$$

because the integrand is an odd function.

Thus, the Fourier Series of $f(x)$ is

$$\frac{2}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x).$$

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**Problem 3**. Consider the function

$$f(x) = 2x, \qquad 0 < x < 1.$$

A. Find the Fourier cosine series of $f(x)$ Hint: you're using the even half-range expansion.

*Answer*:

In this case $(0, L) = (0, 1)$, so $L = 1$. Using the formulas for the even half-range expansion, we get the following.

For $a_0$,

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ &= \int_0^2 2x\,dx \\ &= 1. \end{aligned}$$

For $a_n$ we get

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= 2\int_0^2 2x\cos(n\pi x)\,dx \\ &= -\frac{4}{n^2\pi^2}[1-(-1)^n] \\ &= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^2\pi^2}, & n \text{ odd.} \end{cases} \end{aligned}$$

Using $2k+1,\ k=0,1,2,\ldots$ to range over the odd integers, the Fourier cosine series of $f(x)$ is

$$1-\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x).$$

B. Find the Fourier sine series of $f(x)$. Hint: you're using the odd half-range expansion.

*Answer*:

Using the formulas for the odd half-range expansion, we have

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= 2\int_0^1 2x\sin(n\pi x)\,dx \\
&= \frac{4(-1)^{n+1}}{n\pi}.
\end{aligned}$$

so the Fourier sine series of $f(x)$ is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

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