EXAM

Practice Questions for Exam #3

Math 3350, Spring 2004

April 16, 2004

ANSWERS

Problem 1. Find the general solution **by the method of undetermined coefficients**:

$$(D^2+4)y = x\sin(2x).$$

Answer:
Recall that

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x).$$

Using this formula, we see that $\sin(2x)$ is the imaginary part of e^{2ix}. Thus, $x\sin(2x)$ is the imaginary part of xe^{2ix}. All we need to do is solve the complex equation

$$(D^2+4)y = xe^{2ix} \tag{$\heartsuit$}$$

and then take the imaginary part of our solution.

To solve ($\heartsuit$) move the exponential to the left side and write the equation as

$$e^{-2ix}P(D)y = x,$$

where $P(D) = D^2+4$. Applying the Shifting Rule, we get

$$P(D+2i)[e^{-2ix}y] = x.$$

We set $z = e^{-2ix}y$ and calculate that

$$\begin{aligned} P(D+2i) &= (D+2i)^2+4 \\ &= D^2+4iD-4+4 \\ &= D^2+4iD. \end{aligned}$$

Thus, the equation for z is

$$(D^2+4iD)z = x.$$

Since we can factor out a D, we do so and rewrite the equation as

$$(D+4i)[Dz] = x.$$

Let $w = Dz$, so the equation for w is

$$(D+4i)w = x.$$

This equation has a nonzero constant term in the operator and a polynomial on the right-hand side, so we can solve it by undetermined coefficients. Since the right side is a polynomial of degree 1, we should choose a polynomial of degree 1 for our trial solution, say $w = Ax+B$. Substituting this in the differential equation yields the equation

$$A + 4i[Ax+B] = x.$$

Collecting coefficients in this equation gives

$$4iAx + (A + 4iB) = x.$$

Equating coefficients of powers of x in this equation, we get the equations

$$4iA = 1, \quad A + 4iB = 0.$$

The solution of these equations is

$$A = -\frac{1}{4}i, \quad B = \frac{1}{16}.$$

Thus, we have

$$w = -\frac{1}{4}ix + \frac{1}{16}.$$

Since $w = Dz$, we can find z by integrating w. Thus,

$$z = -\frac{1}{8}ix^2 + \frac{1}{16}x$$

Since $z = e^{-2ix}y$, we have

$$y = \left(-\frac{1}{8}ix^2 + \frac{1}{16}x)\right)e^{2ix}$$

as a particular solution of equation (♡). To find the solution of our original equation, we need to work out the real and imaginary parts of y. We have

$$\begin{aligned} y &= \left(-\frac{1}{8}ix^2 + \frac{1}{16}x)\right)e^{2ix} \\ &= \left(-\frac{1}{8}ix^2 + \frac{1}{16}x\right)(\cos(2x) + i\sin(2x)) \\ &= \frac{1}{16}\cos(2x) + \frac{1}{8}x^2\sin(2x) + i\left[-\frac{1}{8}x^2\cos(2x) + \frac{1}{16}x\sin(2x)\right]. \end{aligned}$$

To get a solution of our original equation, we take the imaginary part of this. Thus, we have

$$y_p = -\frac{1}{8}x^2\cos(2x) + \frac{1}{16}x\sin(2x)$$

as a particular solution of our original equation.

The solution of the homogeneous equation $(D^2+4)y = 0$ is $y_h = C_1\cos(2x) + C_2\sin(2x)$, so the general solution of the equation given in the problem is

$$y = -\frac{1}{8}x^2\cos(2x) + \frac{1}{16}x\sin(2x) + C_1\cos(2x) + C_2\sin(2x).$$

Problem 2. Find the Laplace Transform of the following function:

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t^2 + 2t, & 1 < t < 2 \\ 1, & 2 < t < \infty. \end{cases}$$

Answer:
Using the indicator functions of the intervals, we can write $f(t)$ as

$$f(t) = 0I_{(0,1)}(t) + (t^2 + 2t)I_{(1,2)}(t) + 1I_{(2,\infty)}(t) = (t^2 + 2t)I_{(1,2)}(t) + I_{(2,\infty)}(t).$$

From our table of indicator functions, we have

$$\begin{aligned} I_{(1,2)}(t) &= u(t-1) - u(t-2), \\ I_{(2,\infty)}(t) &= u(t-2). \end{aligned}$$

Plugging this in, we have

$$f(t) = (t^2 + 2t)[u(t-1) - u(t-2)] + u(t-2).$$

Collecting coefficients of the u's, this is

$$f(t) = u(t-1)[t^2 + 2t] + u(t-2)[1 - 2t - t^2]. \tag{2.1}$$

To find the transform of this, we want to use the shifting rule

$$\mathscr{L}[u(t-a)g(t-a)] = e^{-as}G(s), \quad G(s) = \mathscr{L}[g(t)]. \tag{2.2}$$

Consider the first term in (2.1), $u(t-1)[t^2 + 2t]$. To get this to match the left-hand side of (2.2), we must have $u(t-a) = u(t-1)$, so $a = 1$, and $g(t-1) = t^2 + 2t$. From this, we have to figure out what $g(t)$ is. To do this, we substitute $t + 1$ for t, and get

$$\begin{aligned} g(t) &= g(t+1-1) \\ &= (t+1)^2 + 2(t+1) \\ &= t^2 + 2t + 1 + 2t + 2 \\ &= t^2 + 4t + 3. \end{aligned}$$

Then we have

$$G(s) = \mathscr{L}[t^2 + 4t + 3] = \frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}.$$

Applying (2.2) we then get

$$\mathscr{L}[u(t-1)[t^2 + 2t]] = e^{-s}G(s) = e^{-s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right].$$

Similarly, consider the second term in (2.1), $u(t-2)[1-2t-t^2]$. Comparing this with (2.2) we have $a = 2$ and $g(t-2) = 1-2t-t^2$. Thus,

$$\begin{aligned} g(t) &= g(t+2-2) \\ &= 1-2(t+2)-(t+2)^2 \\ &= 1-2t-4-t^2-4t-4 \\ &= -7-8t-t^2. \end{aligned}$$

Then

$$G(s) = \mathscr{L}[-7-8t-t^2] = -\frac{7}{s} - \frac{8}{s^2} - \frac{2}{s^3}.$$

and using (2.2) we get

$$\mathscr{L}[u(t-2)[1-2t-t^2]] = -e^{-2s}\left[\frac{7}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right].$$

Combining these results, we get

$$\mathscr{L}[f(t)] = e^{-s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right] - e^{-2s}\left[\frac{7}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right]$$

Problem 3. Find the Inverse Laplace Transform of the following function:

$$F(s) = \frac{1}{s+1} + e^{-s}\frac{1}{(s+2)^2} + e^{-3s}\frac{1}{s^2+4}$$

Answer:
We know

$$\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}. \tag{3.1}$$

$$\mathscr{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}. \tag{3.2}$$

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{1}{2}\sin(2t). \tag{3.3}$$

We use the shifting rule:

$$\mathscr{L}^{-1}[e^{-as}F(s)] = u(t-a)f(t-a), \quad f(t) = \mathscr{L}^{-1}[F(s)].$$

Applying this rule, and (3.2) we get

$$\mathscr{L}^{-1}\left[e^{-s}\frac{1}{(s+2)^2}\right] = u(t-1)(t-1)e^{-2(t-1)}$$

and from (3.3) we get

$$\mathscr{L}^{-1}\left[e^{-3s}\frac{1}{s^2+4}\right] = \frac{1}{2}u(t-3)\sin(2(t-3)).$$

Thus, we get

$$\mathscr{L}^{-1}[F(s)] = e^{-t} + u(t-1)(t-1)e^{-2(t-1)} + \frac{1}{2}u(t-3)\sin(2(t-3)).$$

Problem 4. Solve the following initial value problem, **using Laplace Transforms**:

$$y'' - 2y' + y = t^2u(t-1) + \delta(t-2), \quad y(0) = 0, \quad y'(0) = 1. \tag{4.1}$$

Answer:
Lets start by finding the Laplace transform of the right-hand side. We know that

$$\mathscr{L}[\delta(t-2)] = e^{-2s}. \tag{4.2}$$

To find the transform of the other term, recall the shifting rule

$$\mathscr{L}[u(t-a)f(t-a)] = e^{-as}F(s), \quad F(s) = \mathscr{L}[f(t)]. \tag{4.3}$$

To match the left-hand side of this with $u(t-1)t^2$ we must have $a = 1$ and $f(t-1) = t^2$. To find $f(t)$, substitute $t+1$ for t in the equation $f(t-1) = t^2$. This gives $f(t) = (t+1)^2 = t^2+2t+1$. Thus, $F(s) = \mathscr{L}[t^2+2t+1] = 2/s^3+2/s^2+1/s$. Hence, by the shifting rule, we have

$$\mathscr{L}[u(t-1)t^2] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] = e^{-s}\frac{s^2+2s+1}{s^3}. \tag{4.4}$$

Now transform both sides of (4.1). The result is

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = e^{-s}\frac{s^2+2s+1}{s^3} + e^{-2s},$$

by (4.2) and (4.4). Putting in the initial conditions and collecting terms gives us

$$(s^2 - 2s + 1)Y(s) - 1 = e^{-s}\frac{s^2+2s+1}{s^3} + e^{-2s}.$$

Solving this for $Y(s)$, we get

$$\begin{aligned} Y(s) &= \frac{1}{s^2-2s+1} + e^{-s}\frac{s^2+2s+1}{s^3(s^2-2s+1)} + e^{-2s}\frac{1}{s^2-2s+1} \\ &= \frac{1}{(s-1)^2} + e^{-s}\frac{s^2+2s+1}{s^3(s-1)^2} + e^{-2s}\frac{1}{(s-1)^2}. \end{aligned}$$

We know, of course,

$$\mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t.$$

To deal with the middle term, we find (by machine) the partial fractions decomposition

$$\frac{s^2+2s+2}{s^3(s-1)^2} = -11\frac{1}{s-1} + 5\frac{1}{(s-1)^2} + 11\frac{1}{s} + \frac{2}{s^3} + 6\frac{1}{s^2}.$$

Thus, the inverse transform is

$$f(t) = \mathscr{L}^{-1}\left[\frac{s^2+2s+2}{s^3(s-1)^2}\right] = -11e^t + 5te^t + 11 + 6t + t^2.$$

By the shifting rule (4.3), we then get

$$\begin{aligned}\mathscr{L}^{-1}\left[e^{-s}\frac{s^2+2s+2}{s^3(s-1)^2}\right] &= u(t-1)f(t-1)\\ &= u(t-1)[(t-1)^2 + 6(t-1) + 11 + 5(t-1)e^{t-1} - 11e^{t-1}].\end{aligned}$$

Similarly,

$$\mathscr{L}^{-1}\left[e^{-2s}\frac{1}{(s-1)^2}\right] = u(t-2)(t-2)e^{t-2}.$$

Thus, finally,

$$y(t) = te^t + u(t-1)[(t-1)^2 + 6(t-1) + 11 + 5(t-1)e^{t-1} - 11e^{t-1}] + u(t-2)(t-2)e^{t-2}.$$

Problem 5. In the following problems, use formulas (1) and (6) from section 5.4.

1. Find the Laplace transform of $f(t) = te^{2t}\cos(t)$.
2. Find the Laplace transform of $f(t) = (e^{2t}-1)/t$.
3. Find the inverse Laplace transform of the function

$$G(s) = \ln\left(\frac{s^2+1}{s^2+4}\right).$$

Answer:
The formulas referred to are

$$\mathscr{L}[tf(t)] = -F'(s), \quad F(s) = \mathscr{L}[f(t)], \tag{5.1}$$

and

$$\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\,d\sigma, \quad F(s) = \mathscr{L}[f(t)], \tag{5.2}$$

provided that $\lim_{t\to 0^+} f(t)/t$ exists.

For the first part, we apply (5.1) with $f(t) = e^{2t}\cos(t)$. From the table, we have

$$\mathscr{L}[e^{2t}\cos(t)] = \frac{s-2}{(s-2)^2+1}.$$

Thus, from (5.1),

$$\begin{aligned}\mathscr{L}[te^{2t}\cos(t)] &= -\frac{d}{ds}\frac{s-2}{(s-2)^2+1}\\ &= \frac{s^2-4s+3}{(s-2)^2+1}.\end{aligned}$$

For the second part, note that $\lim_{t\to 0+}(e^{2t}-1)/t = 2$ (use L'Hôpital's Rule), so we can apply (5.2) with $f(t) = e^{2t}-1$. Then

$$F(s) = \mathscr{L}[e^{2t}-1] = \frac{1}{s-2} - \frac{1}{s}.$$

Thus, by (5.2) we have

$$\begin{aligned}\mathscr{L}\left[\frac{e^{2t}-1}{t}\right] &= \int_s^\infty F(\sigma)\,d\sigma\\ &= \int_s^\infty \left[\frac{1}{\sigma-2} - \frac{1}{\sigma}\right] d\sigma\\ &= \big[\ln(\sigma-2) - \ln(\sigma)\big]_{\sigma=s}^{\sigma=\infty} \qquad \text{(loosely speaking)}\\ &= \ln\left[\frac{\sigma-2}{\sigma}\right]\Bigg|_{\sigma=0}^{\sigma=\infty}\\ &= \ln(1-2/\sigma)\big|_{\sigma=s}^{\sigma=\infty}\\ &= 0 - \ln(1-2/s)\\ &= -\ln(1-2/s)\end{aligned}$$

since

$$\lim_{\sigma\to\infty} \ln(1-2/\sigma) = \ln(1) = 0.$$

For the third part, set

$$G(s) = \ln\left[\frac{s^2+1}{s^2+4}\right].$$

Then

$$G(s) = \ln(s^2+1) - \ln(s^2+4),$$

so we have

$$F(s) = G'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2+4}. \tag{5.3}$$

(this equation defines $F(s)$).

From this we have

$$\int_s^\infty F(\sigma)\, d\sigma = \int_s^\infty G'(\sigma)\, d\sigma = G(\infty) - G(s) = -G(s),$$

(loosely speaking) since $G(\sigma) \to 0$ as $\sigma \to \infty$. Thus, we have

$$G(s) = -\int_s^\infty F(\sigma) d\sigma. \tag{5.4}$$

From the inverse version of (5.2), we then have

$$g(t) = \mathscr{L}^{-1}[G(s)] = -\mathscr{L}^{-1}\left[\int_s^\infty F(\sigma) d\sigma\right] = -\frac{f(t)}{t},$$

where $f(t) = \mathscr{L}^{-1}[F(s)]$. From the table and (5.3),

$$f(t) = 2\sin(t) - 2\sin(2t).$$

Thus, finally, the answer is

$$g(t) = 2\frac{\sin(t) - \sin(2t)}{t}.$$

You can verify that $\lim_{t\to 0^+} g(t)$ exists by L'Hôpital's Rule.

Problem 6. Find the convolution $t^2 * t$ directly from the definition.

Answer:

The definition of convolution is

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

Let $f(t) = t^2$ and $g(t) = t$. Plugging into the formula above, we have

$$\begin{aligned} (f*g((t) &= \int_0^t \tau^2(t-\tau)\, d\tau \\ &= \int_0^t (t\tau^2 - \tau^3)\, d\tau \\ &= \left[\frac{1}{3}t\tau^3 - \frac{1}{4}\tau^4\right]_{\tau=0}^{\tau=t} \\ &= \frac{1}{3}t^4 - \frac{1}{4}t^4 - [0-0] \\ &= \frac{1}{12}t^4. \end{aligned}$$

Problem 7. Find the convolution $\cos(2t) * \sin(t)$ **using Laplace transforms**.

Answer:
We want to use the formula

$$\mathscr{L}[(f*g)(t)] = F(s)G(s). \tag{7.1}$$

We know, of course, that

$$\begin{aligned}\mathscr{L}[\cos(2t)] &= \frac{s}{s^2+4}\\ \mathscr{L}[\sin(t)] &= \frac{1}{s^2+1}.\end{aligned}$$

Thus, by (7.1),

$$\mathscr{L}[\cos(2t) * \sin(t)] = \frac{s}{s^2+4}\frac{1}{s^2+1} = \frac{s}{(s^2+4)(s^2+1)}.$$

All we have to do is take the inverse transform of the right-hand side. The partial fractions decomposition is (by machine)

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}.$$

Thus, we have

$$\begin{aligned}\cos(2t) * \sin(t) &= \mathscr{L}^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right]\\ &= \frac{1}{3}\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{3}\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right]\\ &= \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t).\end{aligned}$$

Problem 8. Express the solution of the following initial value problem using a convolution integral.

$$y'' - 2y' + y = r(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Answer:
Take the Laplace Transform of both sides of the equation. This gives

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = R(s),$$

where $R(s) = \mathscr{L}[r(t)]$. Plugging in the initial conditions and simplifying we get

$$(s^2 - 2s + 1)Y(s) = 1 + R(s).$$

Solving this equation for $Y(s)$ gives us

$$Y(s) = \frac{1}{s^2 - 2s + 1} + \frac{R(s)}{s^2 - 2s + 1} = \frac{1}{(s-1)^2} + \frac{R(s)}{(s-1)^2}.$$

As we know,

$$\mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t.$$

To take the inverse transform of the other term, we use the formula

$$\mathscr{L}^{-1}[F(s)G(s)] = (f * g)(t).$$

Thus,

$$\mathscr{L}^{-1}\left[R(s)\frac{1}{(s-1)^2}\right] = r(t) * te^t.$$

From the definition of convolution,

$$r(t) * te^t = \int_0^t r(\tau)(t-\tau)e^{t-\tau}\, d\tau.$$

Putting all this together, the solution of our initial value problem is

$$y(t) = \int_0^t (t-\tau)e^{t-\tau} r(\tau)\, d\tau + te^t.$$
