
PROBLEM SET

Practice Problems for Exam #2

Math 2350, Fall 2004

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ANSWERS

Problem 1. Consider the function $f(x,y) = xy^2\sin(x^2y)$. Find the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yx}$ and f_{yy}.

Answer:
For the first order derivatives, we have

$$f_x(x,y) = y^2\sin(x^2y) + xy^2\cos(x^2y)2xy = y^2\sin(x^2y) + 2x^2y^3\cos(x^2y),$$

and

$$f_y(x,y) = 2xy\sin(x^2y) + xy^2\cos(x^2y)x^2 = 2xy\sin(x^2y) + x^3y^2\cos(x^2y)..$$

For the second order derivatives, we have

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}[y^2\sin(x^2y) + 2x^2y^3\cos(x^2y)] \\
&= y^2\cos(x^2y)2xy + 4xy^2\cos(x^2y) - 2x^2y^3\sin(x^2y)2xy \\
&= 2xy^3\cos(x^2y) + 4xy^3\cos(x^2y) - 4x^3y^4\sin(x^2y)
\end{aligned}$$

$$\begin{aligned}
f_{xy}(x,y) &= \frac{\partial}{\partial y}[y^2\sin(x^2y) + 2x^2y^3\cos(x^2y)] \\
&= 2y\sin(x^2y) + y^2\cos(x^2y)x^2 + 6x^2y^2\cos(x^2y) - 2x^2y^3\sin(x^2y)x^2 \\
&= 2y\sin(x^2y) + 7x^2y^2\cos(x^2y) - 2x^4y^3\sin(x^2y)
\end{aligned}$$

$$\begin{aligned}
f_{yx}(x,y) &= \frac{\partial}{\partial x}[2xy\sin(x^2y) + x^3y^2\cos(x^2y)] \\
&= 2y\sin(x^2y) + 2xy\cos(x^2y)2xy + 3x^2y^2\cos(x^2y) - x^3y^2\sin(x^2y)2xy \\
&= 2y\sin(x^2y) + 7x^2y^2\cos(x^2y) - 2x^4y^3\sin(x^2y)
\end{aligned}$$

$$\begin{aligned}
f_{yy}(x,y) &= \frac{\partial}{\partial y}[2xy\sin(x^2y) + x^3y^2\cos(x^2y)] \\
&= 2x\sin(x^2y) + 2xy\cos(x^2y)x^2 + 2x^3y\cos(x^2y) - x^3y^2\sin(x^2y)x^2 \\
&= 2x\sin(x^2y) + 4x^3y\cos(x^2y) - x^4y^2\sin(x^2y).
\end{aligned}$$

Problem 2. **Use the chain rule for partial derivatives** to find $\partial r/\partial x$ and $\partial r/\partial y$ where $r = uv - u^2 - v^2$, $u = x^2y$ and $v = 2x - y$. Express your answer in terms of x and y.

Answer:
We have

$$\begin{aligned}\frac{\partial r}{\partial u} &= v - 2u\\ \frac{\partial r}{\partial v} &= u - 2v\\ \frac{\partial u}{\partial x} &= 2xy\\ \frac{\partial u}{\partial y} &= x^2\\ \frac{\partial v}{\partial x} &= 2\\ \frac{\partial v}{\partial y} &= -1\end{aligned}$$

By the chain rule,

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x}\\ &= (v - 2u)2xy + (u - 2v)2\\ &= (2x - y - 2x^2y)2xy + (x^2y - 4x + 2y)2\\ &= 4x^2y - 2xy^3 - 4x^3y^2 + 2x^2y - 8x + 4y\\ &= 6x^2y - 2xy^3 - 4x^3y^2 - 8x + 4y\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{\partial r}{\partial y} &= \frac{\partial r}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial y}\\ &= (v - 2u)x^2 + (u - 2v)(-1)\\ &= (2x - y - 2x^2y)x^2 - (x^2y - 4x + 2y)\\ &= 2x^3 - x^2y - 2x^4y - x^2y + 4x - 2y\\ &= 2x^3 - 2x^2y - 2x^4y + 4x - 2y.\end{aligned}$$

Problem 3. Suppose that $p = f(u, v, w)$, $u = g(x, y)$, $v = h(y, z)$ and $w = k(x, y, z)$. Write the chain rule formals for the partial derivatives of p with respect to x, y and z.

Answer:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial x}\\ \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial y}\\ \frac{\partial p}{\partial z} &= \frac{\partial p}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial z}.\end{aligned}$$

Problem 4. A box is measured to have a length of 15 inches, a width of 10 inches and a depth of 5 inches. From these measurements, we would calculate the volume to be 750 cubic inches. If there is a possible error of up to 0.25 inch in the measurements, use differentials to estimate the maximum possible error in the calculated volume.

Answer:
Let x, y and z denote the length, width and depth of the box. Then the volume is given by $V = xyz$. The total differential of the volume is

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz \\ &= yz\,dx + xz\,dy + xy\,dz. \end{aligned}$$

To approximate the error in the volume, we plug in $x = 15$, $y = 10$, $z = 5$ and $dx = dy = dz = 0.25$. Thus,

$$\begin{aligned} \text{Error in Volume} &\approx dV \\ &= (10)(5)(0.25) + (15)(5)(0.25) + (15)(10)(0.25) \\ &= 68.75 \end{aligned}$$

cubic inches.

Problem 5. A rectangular block of ice is melting in the sun. At a certain instant the block has a height of 5 inches, a length of 10 inches, and a width of 12 inches, and each of the dimensions is decreasing at a rate of 2 inches per hour. How fast is the volume of the block changing?

Answer:
Let x be the height of the box, y the length and z the width so the volume of the box is $V = xyz$. By the chain rule

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} \\ &= yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt} \end{aligned}$$

At the given instant, we have

$$\begin{gathered} x = 5, \qquad y = 10, \qquad z = 12, \\ \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = -2 \end{gathered}$$

(minus since the dimensions are decreasing). Thus, at this instant,

$$\begin{aligned}\frac{dV}{dt} &= (10)(12)(-2) + (5)(12)(-2) + (5)(10)(-2) \\ &= -460,\end{aligned}$$

so the volume is decreasing at a rate of 460 cubic inches per hour.

Problem 6. Find the directional derivative of the function $f(x, y) = x^2 + 2xy$ at the point $(1, 2)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

Answer:
The gradient of f is given by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (2x + 2y)\mathbf{i} + 2x\mathbf{j}$$

and so

$$\nabla f(1, 2) = 6\mathbf{i} + 2\mathbf{j}.$$

The length of the vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5,$$

so a unit vector in the direction of v is

$$\mathbf{u} = \frac{1}{5}\mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

Thus, the directional derivative we want is

$$\begin{aligned}D_{\mathbf{u}}f(1, 2) &= \nabla f(1, 2) \cdot \mathbf{u} \\ &= (6\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) \\ &= 26/5.\end{aligned}$$

Problem 7. Suppose that the temperature at a point (x, y) in the plane is given by $f(x, y) = x^2 + xy + y^2$. A bug is walking around in the plane with a speed of 5 units per minute.

A. At a certain instant the bug is at the point $(2, 1)$ and has velocity vector $\mathbf{v} = (3, -4)$. What rate of change of temperature is the bug experiencing at this instant?

Answer:

If the bug's position at time t is $(x(t), y(t))$, the bug's temperature at time t is $T(t) = f(x(t), y(t))$. The rate of change of the bug's temperature is

$$\frac{dT}{dt}(t) = \nabla f(x(t), y(t)) \cdot \mathbf{v}(t),$$

where $\mathbf{v}(t)$ is the bug's velocity vector. We have

$$\nabla f(x, y) = (2x + y, x + 2y).$$

At our instant, call it $t = t_0$, we have $(x(t_0), y(t_0)) = (2, 1)$ and $\mathbf{v}(t_0) = (3, -4)$. Thus, we have

$$\begin{aligned}\frac{dT}{dt}(t_0) &= \nabla f(x(t_0), y(t_0)) \cdot \mathbf{v}(t_0) \\ &= \nabla f(2, 1) \cdot (3, -4) \\ &= (5, 4) \cdot (3, -4) \\ &= 15 - 16 \\ &= -1.\end{aligned}$$

B. Suppose the bug is at $(2, 1)$. If the bug wants to cool off, what direction should he travel (at speed 5) to get fastest decrease in temperature? What is this rate of change of temperature?

Answer:

The direction of fastest decrease is the direction of $-\nabla f(2, 1)$. The vector of length 5 in this direction is

$$\mathbf{v} = -\frac{5}{\|\nabla f(2, 1)\|} \nabla f(2, 1),$$

so this should be the bug's velocity vector. The rate of change of temperature will be

$$\begin{aligned}\frac{dT}{dt} &= \nabla f(2, 1) \cdot \mathbf{v} \\ &= \nabla f(2, 1) \cdot \left[-\frac{5}{\|\nabla f(2, 1)\|} \nabla f(2, 1) \right] \\ &= -5 \frac{\|\nabla f(2, 1)\|^2}{\|\nabla f(2, 1)\|} \\ &= -5 \|\nabla f(2, 1)\| \\ &= -5\sqrt{41}.\end{aligned}$$

Problem 8. Find the equation of the tangent plane to the cone $z^2 = x^2 + y^2$ at the point $P = (4, -3, 5)$. Find parametric equations for the normal line to the surface at this point.

Answer:
The cone is the level surface

$$g(x, y, z) = x^2 + y^2 - z^2.$$

The gradient of this function is

$$\nabla g(x, y, z) = (2x, 2y, -2z).$$

Since ∇g is orthogonal to the level surfaces of g, the vector

$$\mathbf{N} = \nabla g(4, -3, 5) = (8, -6, -10)$$

is perpendicular to the surface (i.e., perpendicular to the tangent plane of the surface) at P. Hence the tangent plane is the plane that passes through the point with position vector $\mathbf{x}_0 = (4, -3, 5)$ and is perpendicular to $\mathbf{N}$. The vector form of the equation is

$$\mathbf{N} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

where $\mathbf{x} = (x, y, z)$. Thus, we have

$$\begin{aligned} 0 &= \mathbf{N} \cdot (\mathbf{x} - \mathbf{x}_0) \\ &= \mathbf{N} \cdot [(x, y, z) - (4, -3, 5)] \\ &= \mathbf{N} \cdot (x - 4, y + 3, z - 5) \\ &= (8, -6, -10) \cdot (x - 4, y + 3, z - 5) \\ &= 8(x - 4) - 6(y + 3) - 10(z - 5). \end{aligned}$$

Thus, the equation of the tangent plane is

$$8(x - 4) - 6(y + 3) - 10(z - 5) = 0,$$

which can be simplified to standard form as

$$4x - 3y - 5z = 0.$$

The normal line passes through P and is parallel to $\nabla g(P) = \mathbf{N}$, so the vector form of the parametrization is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{N} = (4, -3, 5) + t(8, -6, -10) = (4 + 8t, -3 - 6t, 5 - 10t),$$

so the parametric equations are

$$\begin{aligned} x &= 4 + 8t \\ y &= -3 - 6t \\ z &= 5 - 10t \end{aligned}$$

Problem 9. Locate an classify the critical points of the function

$$f(x, y) = x^4 + 4xy + y^4.$$

Answer:
We can calculate the partial derivatives as follows

$$\begin{aligned} f_x(x, y) &= 4x^3 + 4y \\ f_y(x, y) &= 4x + 4y^3 \\ f_{xx}(x, y) &= 12x^2 \\ f_{xy}(x, y) &= 4 \\ f_{yy}(x, y) &= 12y^2. \end{aligned}$$

To find the critical points we want to solve the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$. We can divide both equations by 4, so we want to solve the system of equations

$$x^3 + y = 0 \tag{9.1}$$

$$x + y^3 = 0 \tag{9.2}$$

From (9.1) we have

$$y = -x^3. \tag{9.3}$$

Plugging this into (9.2) gives $x + (-x^3)^3 = 0$ or $x - x^9 = 0$. This factors as $x(1 - x^8) = 0$. The solutions of this equation are $x = 0$ and $x = \pm\sqrt[8]{1} = \pm 1$. Plugging these values of x into (9.3) we find three critical points $(0, 0)$, $(1, -1)$ and (-1,1).

Consider the critical point $(0, 0)$. At this point, the value of the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ is $-16 < 0$, so $(0, 0)$ is a saddle point.

At $(1, -1)$ the value of the discriminant is $D = 144 - 16 > 0$, so we have a relative extrema. We have $f_{xx} > 0$ at this point, so the graph is concave up in the x-direction. Thus $(1, -1)$ is a relative min.

Similarly, we have $D = 144 - 16 > 0$ and $f_{xx} > 0$ at $(-1, 1)$, so $(-1, 1)$ is a relative min.

Problem 10. Find the absolute max and min of the function $f(x, y) = xy$ on the disk $x^2 + y^2 \leq 2$. You can parametrize the boundary, or use Lagrange multipliers.

Answer:
The region is the disk of radius $\sqrt{2}$ centered at the origin. First we find the critical points in the interior of the region. The partials are

$$\begin{aligned} f_x(x, y) &= y \\ f_y(x, y) &= x. \end{aligned}$$

Setting these equal to zero gives the system of equations

$$\begin{aligned} y &= 0 \\ x &= 0. \end{aligned}$$

Thus, we get one critical point $(0,0)$ of f which is in the interior of the region. We have $f(0,0) = 0$.

Now we have to find the max and min of f on the boundary circle $x^2+y^2 = 2$. You could do this by parameterizing the circle by $x = \sqrt{2}\cos(\theta)$ and $y = \sqrt{2}\sin(\theta)$. I'll do it by Lagrange multipliers.

Thus we want to maximize and minimize $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2+y^2 = 2$. The Lagrange multiplier equations are $\nabla f = \lambda \nabla g$ and $g = 2$ in vector form. In order words

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g &= 2. \end{aligned}$$

Plugging in the formulas we get this system

$$y = 2\lambda x \tag{10.1}$$

$$x = 2\lambda y \tag{10.2}$$

$$x^2 + y^2 = 2. \tag{10.3}$$

Plugging (10.1) into (10.2) gives $x = 4\lambda^2 x$ or

$$(4\lambda^2 - 1)x = 0. \tag{10.4}$$

First, consider the case where $4\lambda^2 - 1 \neq 0$. Then (10.4) implies that $x = 0$. But then (10.1) implies that $y = 0$. The point $(0,0)$ does not satisfy the constraint equation (10.3), so there is no solution of the system with $4\lambda^2 - 1 \neq 0$.

Thus, we must have $4\lambda^2 - 1 = 0$, i.e., $\lambda = \pm 1/2$.

Consider first the case $\lambda = 1/2$. Then equation (10.1) and equation (10.2) become $y = x$. Substituting this into the constraint (10.3) gives $2x^2 = 2$, so $x = \pm 1$. Since $y = x$, we get two critical points $(1,1)$ and $(-1,-1)$. The value of the function at these points is $f = 1$.

In the case $\lambda = -1/2$, (10.1) gives $y = -x$. Plugging this into the constraint equation again gives $2x^2 = 2$, so $x = \pm 1$. Thus, we get two critical points $(1,-1)$ and $(-1,1)$. The value of the function at these points is $f = -1$.

At the points that are candidates for the absolute max and min of f, we have values -1, 0, 1. Thus, the absolute min is of f on the disc is -1, which occurs at the points $(1,-1)$ and $(-1,1)$. The absolute max of f on the disk is 1, which occurs at the points $(1,1)$ and $(-1,-1)$.

Problem 11. Find the absolute max and min of the function $f(x, y, z) = xyz^2$ on the spherical surface $x^2 + y^2 + z^2 = 128$. (The absolute max and min exist because the surface is closed and bounded.)

Answer:
The problem is to extermize the function $f(x, y, z) = xyz^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 128$. The Lagrange multiplier equations are $\nabla f = \lambda \nabla g$ and $g = 128$. Plugging in the formulas for the partial derivatives this is equivalent to

$$yz^2 = 2\lambda x \tag{11.1}$$

$$xz^2 = 2\lambda y \tag{11.2}$$

$$2xyz = 2\lambda z \implies xyz = \lambda z. \tag{11.3}$$

$$x^2 + y^2 + z^2 = 128 \tag{11.4}$$

Now multiply (11.1) by x, (11.2) by y, and (11.3) by z to get

$$xyz^2 = 2\lambda x^2 \tag{11.5}$$

$$xyz^2 = 2\lambda y^2 \tag{11.6}$$

$$xyz^2 = \lambda z^2. \tag{11.7}$$

Thus, we have

$$2\lambda x^2 = 2\lambda y^2 = \lambda z^2. \tag{11.8}$$

Consider first the case $\lambda = 0$. Then, from (11.1) through (11.3) we must have $xz^2 = 0$, $yz^2 = 0$ and $xyz = 0$. These equations are satisfied if $z = 0$. Thus, all the points on the circle $x^2 + y^2 = 128$ in the plane $z = 0$ are critical points. The value of the function f at these points is zero. If $z \neq 0$, we must have $x = y = 0$. From the constraint, this implies that $z^2 = 128$, so $z = \pm 8\sqrt{2}$. The function is zero at these points.

If $\lambda \neq 0$, the we can divide (11.8) by λ to get

$$2x^2 = 2y^2 = z^2. \tag{11.9}$$

Multiplying the constraint equation by 2 gives

$$2x^2 + 2y^2 + 2z^2 = 256$$

Substituting z^2 for $2z^2$ and $2y^2$ gives $4z^2 = 256$, so $z = \pm 8$. We then have $2x^2 = z^2 = 64$, so $x = \pm\sqrt{32} = \pm 4\sqrt{2}$. Similarly, $y = \pm 4\sqrt{2}$. Thus, we get 8 critical points $(\pm 4\sqrt{2}, \pm 4\sqrt{2}, \pm 8)$. The values of the function at these points are ± 256.

Thus, the absolute min is -256 and the absolute max is 256. I'll leave it to the reader to list the points at which these occur.

Problem 12. Consider the curve C formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Find the points on the curve that are closest and farthest from the origin. (The curve is closed and bounded.)

Answer:
I'll use the usual trick of extermizing the square of the distance from the origin. Thus, our problem is to extermize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 = 1$ and $g_2(x, y, z) = x + y + z = 1$. The Lagrange multiplier equations are

$$\begin{aligned} \nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 &= 1 \\ g_2 &= 1. \end{aligned}$$

Writing these out gives the system of equations

$$\begin{aligned} 2x &= 2\lambda_1 x + \lambda_2 \\ 2y &= 2\lambda y + \lambda_2 \\ 2z &= \lambda_2 \\ x^2 + y^2 &= 1 \\ x + y + z &= 1 \end{aligned}$$

These can be rewritten as

$$2(1 - \lambda_1)x = \lambda_2 \tag{12.1}$$
$$2(1 - \lambda_1)y = \lambda_2 \tag{12.2}$$
$$2z = \lambda_2 \tag{12.3}$$
$$x^2 + y^2 = 1 \tag{12.4}$$
$$x + y + z = 1 \tag{12.5}$$

Consider first the case $\lambda_1 = 1$. Then (12.1) and (12.2) imply that $\lambda_2 = 0$. Plugging this into (12.3) gives $z = 0$. Plugging $z = 0$ into (12.4) and (12.5) gives the equations $x^2 + y^2 = 1$ and $x + y = 1$. Substitute $y = 1 - x$ into $x^2 + y^2 = 1$. Then we have $x^2 + (1 - x)^2 = x^2 + 1 - 2x + x^2 = 1$, which reduces to $x(x - 1) = 0$, so we have solutions $x = 0$ and $x = 1$. Using $x + y = 1$ and $z = 0$ we get two critical points $(1, 0, 0)$ and $(0, 1, 0)$. The value of f at these points is 1, so the distance of these points from the origin is 1.

If $\lambda \neq 1$ then (12.1) and (12.2) imply that $x = y$. Plugging this into (12.4) we get $2x^2 = 1$, so $x = \pm\sqrt{2}/2$.

If $x = \sqrt{2}/2$, we have $y = x = \sqrt{2}/2$ and we can use (12.5) to solve for z. This gives us a critical point $(\sqrt{2}/2, \sqrt{2}/2, 1 - \sqrt{2})$. The value of f at this point is $1 + (1 - \sqrt{z})^2 \approx 1.1716$.

If $x = -\sqrt{2}/2$ we have $y = x = -\sqrt{2}/2$ and we can use (12.5) to solve for z. This gives us a critical point $(-\sqrt{2}/2, -\sqrt{2}/2, 1 + \sqrt{2})$. The value of f at this point is $1 + (1 + \sqrt{2})^2 \approx 6.8284$.

The smallest value of f at a critical point is 1. Thus, the points on C closest to the origin are $(1, 0, 0)$ and $(0, 1, 0)$ at a distance of 1.

The largest value of f at a critical point is $1 + (1 + \sqrt{2})^2$. Thus, the point on C that is farthest from the origin is $(-\sqrt{2}/2, -\sqrt{2}/2, 1 + \sqrt{2})$ at a distance of $\sqrt{1 + (1 + \sqrt{2})^2} \approx 2.6131$.
