PROBLEM SET

Practice Problems for Exam #1

Math 1352, Fall 2004

Oct. 1, 2004

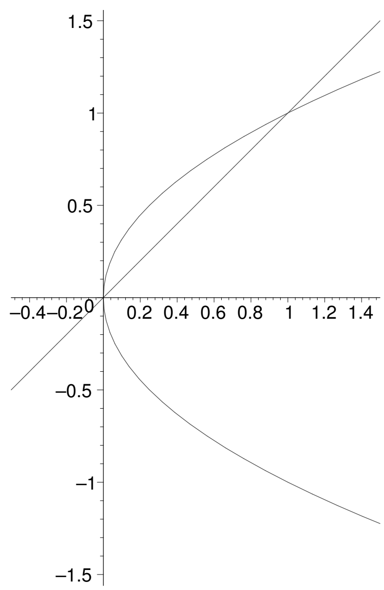
ANSWERS

Problem 1.vLet R be the region bounded by the curves $x = y^2$ and $y = x$.

A. Find the volume of the solid generated by revolving the region R around the x-axis.

Answer:

Here's the picture of the region R:



The intersections of the two curves are at $(0, 0)$ and $(1, 1)$.

For revolution about the x-axis, we use the method of washers. The outside radius is the top curve x and the inside radius is the bottom curve x^2. Thus, the volume V is given by

$$\begin{aligned} V &= \pi \int_a^b \left\{ [\text{outside radius}]^2 - [\text{inside radius}]^2 \right\} dx \\ &= \pi \int_0^1 \left\{ x^2 - [x^2]^2 \right\} dx \\ &= \pi \int_0^1 (x^2 - x^4)\, dx \\ &= \pi [x^3/3 - x^5/5]_0^1 \\ &= \pi(1/3 - 1/5) \\ &= \frac{2\pi}{15} \end{aligned}$$

B. Find the volume of the solid generated by revolving the region R around the y-axis.

Answer:

First Solution: Use the method of shells, integrating in the x-direction. The volume V is then given by

$$\begin{aligned} V &= 2\pi \int_a^b (\text{radius})(\text{height})\, dx \\ &= 2\pi \int_0^1 (x)(x - x^2)\, dx \\ &= 2\pi \int_0^1 (x^2 - x^3)\, dx \\ &= 2\pi[x^3/3 - x^4/4]_0^1 \\ &= 2\pi[1/3 - 1/4] \\ &= \pi/6. \end{aligned}$$

Second Solution: Use the method of washers, integrating in the y-direction. The outside radius is on the parabola. Solving the equation $y = x^2$ for x gives us $x = \sqrt{y}$ for the outside radius. The inside radius is on the line, and so the inside radius is $x = y$. For the region R, y ranges from 0 to 1. Thus, the volume is given by

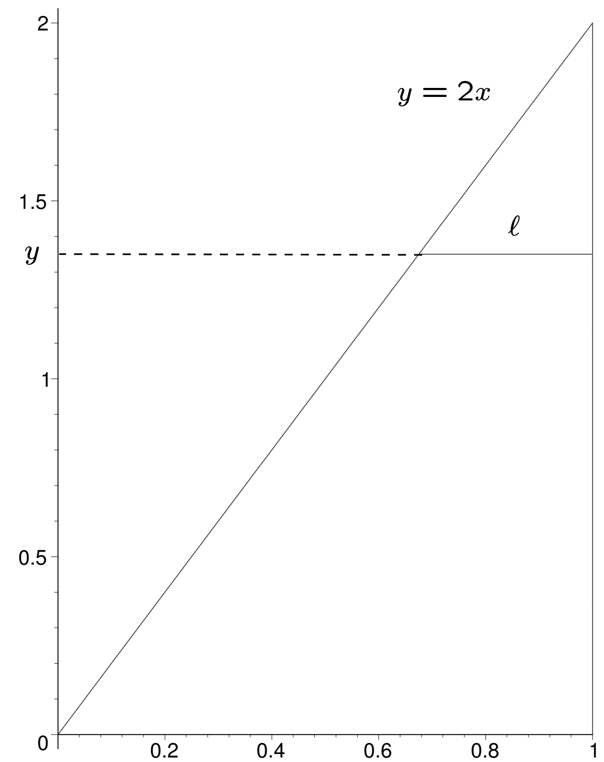
$$\begin{aligned} V &= \pi \int_0^1 \left\{[\sqrt{y}]^2 - y^2\right\} l\, dy \\ &= \pi \int_0^1 (y - y^2)\, dy \\ &= \pi[y^2/2 - y^3/3]_0^1 \\ &= \pi/6. \end{aligned}$$

Problem 2.

The base of a solid is the region in the xy-plane bounded by the lines $y = 2x$ and $x = 1$. The cross sections of the solid perpendicular to the y-axis are squares. Find the volume of the solid.

Answer:

The picture looks like this.



If we take a cross section at y, the base of the square cross section is the segment labeled ℓ. To find the length ℓ of this segment, note that the left-hand endpoint is one the line $y = 2x$, so its x-coordinate is $y/2$. The x-coordinate of the right-hand endpoint is at $x = 1$, so $\ell = 1 - y/2$. The area of the cross section at y is thus $A(y) = (1 - y/2)^2$. We want the cross sections to sweep out the solid, so y should range from 0 to 2. Thus, we have

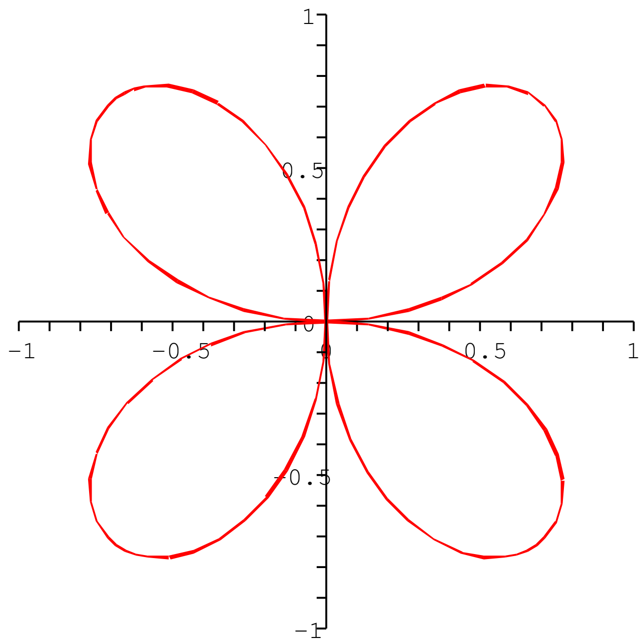
$$V = \int_0^2 A(y)\,dy = \int_0^2 (1 - y/2)^2\,dy.$$

To do this integral, make the substitution $u = 1 - y/2$. Then $dy = -2\,du$. When $y = 0$ we have $u = 1$ and when $y = 2$ we have $u = 0$. Thus, we get

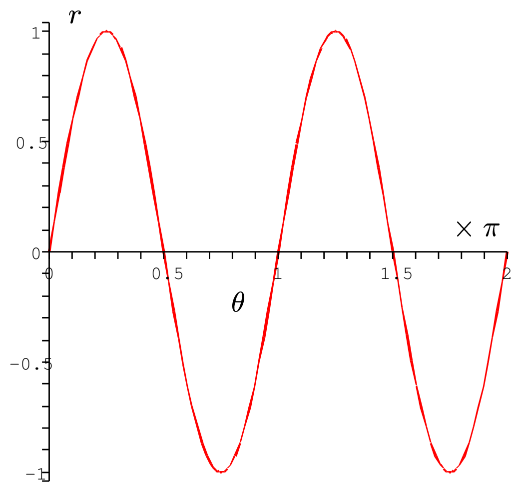
$$\begin{aligned} V &= \int_1^0 u^2(-2\,du) \\ &= 2\int_0^1 u^2\,du \\ &= \frac{2}{3}u^3\Big|_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

Problem 3. Find the area of one leaf of the four-leaf rose whose polar equation is $r = \sin(2\theta)$.

Answer:
The polar graph of the equation $r = \sin(2\theta)$ looks like this:



Lets find the area of the petal in the first quadrant. To see what range of θ is required to draw this petal, consider the rectangular coordinate graph of $r = \sin(2\theta)$, which looks like this:



[Note that the θ-axis is marked in units of π].

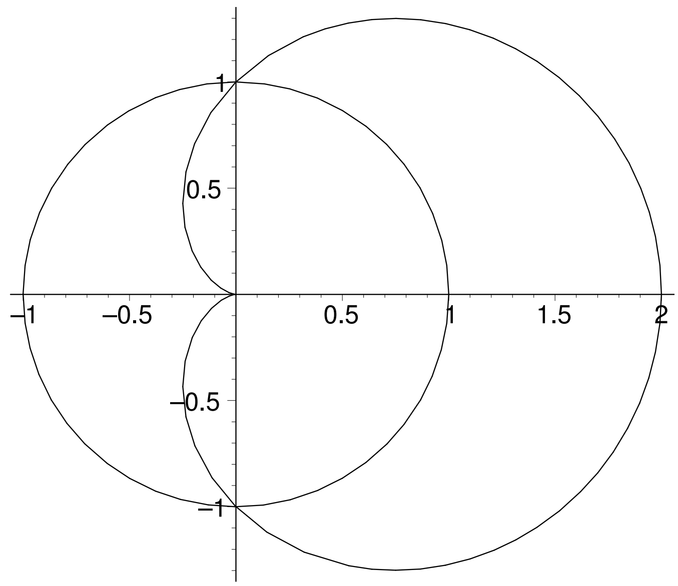
From this graph, we see that the petal in the first quadrant is drawn as θ goes from 0 to $\pi/2$. Thus, the area A is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} \sin^2(2\theta)\, d\theta. \\ &= \frac{1}{2}\int_{0}^{\pi/2} \left[\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right] \, d\theta \qquad \text{(double angle formula)} \\ &= \frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{8}\sin(4\theta)\right]_0^{\pi/2} \\ &= \frac{1}{2}\left\{\left[\frac{\pi}{4} - \frac{1}{8}\sin\left(4\frac{\pi}{2}\right)\right] - \left[0 - \sin(0)\right]\right\} \\ &= \frac{\pi}{8}. \end{aligned}$$

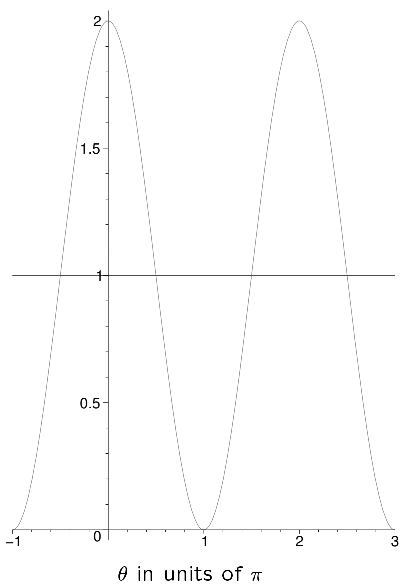
Problem 4.

Find the area of the region that is inside the cardioid $r = 1 + \cos(\theta)$ and outside the circle $r = 1$.

Answer:



The rectangular plot of $r = 1 + cos(\theta)$ looks like this:



From this we can see that if we start at $\theta = -\pi/2$ and proceed to $\pi/2$, the part of the cardioid that is outside the circle will be drawn. For a fixed value of θ in this range, the outside curve will be $r = 1 + \cos(\theta)$ and the inside curve is

$r = 1$, thus the area is given by

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\{[1+\cos(\theta)]^2 - 1^2\}\,d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2}[1+2\cos(\theta)+\cos^2(\theta)-1]\,d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2}[2\cos(\theta)+\cos^2(\theta)]\,d\theta \\
&= \frac{2}{2}\int_{0}^{\pi/2}[2\cos(\theta)+\cos^2(\theta)]\,d\theta \qquad \text{(cos is even)} \\
&= 2\int_0^{\pi/2}\cos(\theta)\,d\theta + \int_0^{\pi/2}\cos^2(\theta)\,d\theta \qquad (4.1)
\end{aligned}
$$

For the first integral in (4.1) we get

$$
\int_0^{\pi/2}\cos(\theta)\,d\theta = \sin(\theta)\Big|_0^{\pi/2} = \sin(\pi/2)-\sin(0) = 1-0 = 1 \qquad (4.2)
$$

For the second integral in (4.1) we have

$$
\begin{aligned}
\int_0^{\pi/2}\cos^2(\theta)\,d\theta &= \int_0^{\pi/2}\frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2}1\,d\theta + \frac{1}{2}\int_0^{\pi/2}\cos(2\theta)\,d\theta\,d\theta \\
&= \frac{\pi}{4} + \frac{1}{4}\Big[\sin(2\theta)\Big]_0^{\pi/2} \\
&= \frac{\pi}{4} + \frac{1}{4}[\sin(\pi)-\sin(0)] \\
&= \frac{\pi}{4} + \frac{1}{4}[0-0] \\
&= \frac{\pi}{4} \qquad (4.3)
\end{aligned}
$$

Plugging the results of (4.2) and (4.3) into (4.1), we get

$$A = 2 + \frac{\pi}{4}.$$

Problem 5.

Find the arc length of the graph of

$$f(x) = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}$$

on the interval from [1.2]. (This is Problem 9 on page 393 of the book.)

Answer:
The formula for the arc length of a graph is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

In our case we have

$$f'(x) = x^2 - \frac{1}{4}x^{-2} = x^2 - \frac{1}{4x^2}.$$

This gives us

$$\begin{aligned}
[f'(x)]^2 &= \left[x^2 - \frac{1}{4x^2}\right]^2 \\
&= [x^2]^2 + 2[x^2]\left[\frac{-1}{4x^2}\right] + \left[-\frac{1}{4x^2}\right]^2 \\
&= x^4 - \frac{1}{2} + \frac{1}{16x^4}
\end{aligned}$$

We can then compute that

$$\begin{aligned}
1 + [f'(x)]^2 &= 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4} \\
&= \frac{1}{2} + x^4 + \frac{1}{16x^4} \\
&= \frac{8x^4 + 16x^8 + 1}{16x^4} \\
&= \frac{(4x^4 + 1)^2}{16x^4}.
\end{aligned}$$

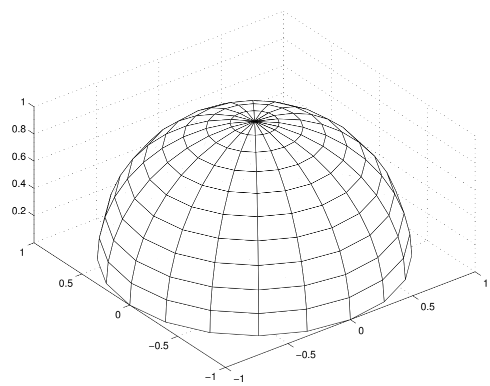
Thus, we get

$$\sqrt{1 + [f'(x)]^2} = \frac{4x^4 + 1}{4x^2} = x^2 + \frac{1}{4x^2}.$$

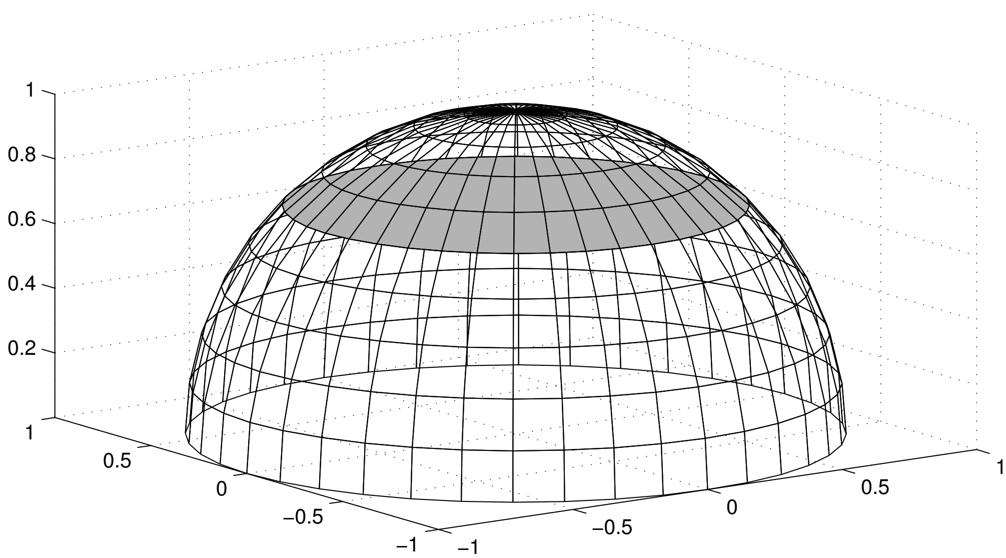
So, finally, we have

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx \\
&= \int_1^2 (x^2 + 4x^{-2})\, dx \\
&= \left[\frac{1}{3}x^3 - \frac{4}{x}\right]_1^2 \\
&= \frac{59}{24}
\end{aligned}$$

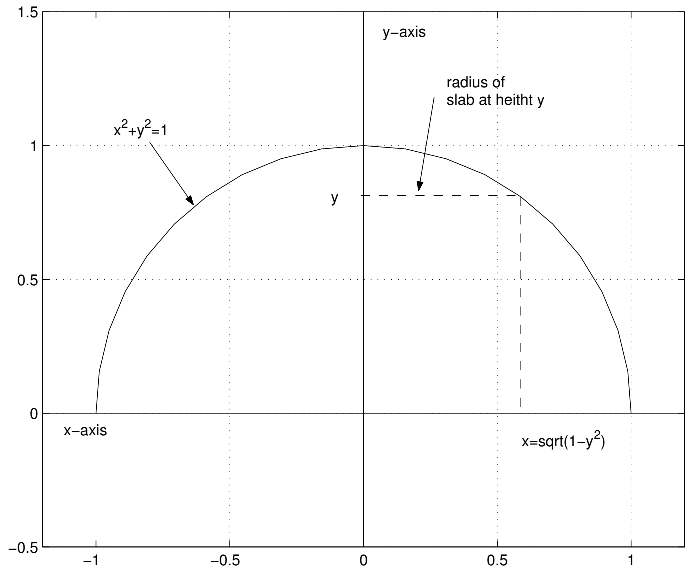
Problem 6. A tank has the shape of a hemisphere (see picture) of radius 1 meter. The tank is full of water, which weights $9800\,\mathrm{N/m^3}$. How much work is required to pump all of the water to a point one meter above the top of the tank?



Answer:
Put in the y-axis running upward from ground level. Consider a slab of the water in the tank at height y and with thickness dy–for example, the shaded slab in the picture below.



We need to calculate the volume of this slab, using the formula for the volume of a cylinder. The height of the slab is dy. To figure out the radius of the slab, take a cross-section of the picture, as in the next figure.



The cross section of the tank is a semi-circle of the circle of radius one. The equation of the circle of radius one centered at the origin is $x^2 + y^2 = 1$. Solving this for x yields $x = \pm\sqrt{1 - y^2}$. Thus, as we see from the figure, the radius of the slab is $\sqrt{1 - y^2}$. Thus, the volume of the slab is given by

$$\pi(\sqrt{1 - y^2})^2 \, dy = \pi(1 - y^2) \, dy.$$

Thus, the weight of the slab is

$$\pi w(1 - y^2) \, dy,$$

where $w = 9800$ is the weight density of water. The slab is a height y and we need to lift it to $y = 2$ (one meter above the top of the tank at $y = 1$), so the distance to lift the slab is $2 - y$. Thus, the total work for lifting this slab is

$$(2 - y) \cdot (\pi w(1 - y^2) \, dy) = \pi w(2 - y)(1 - y^2) \, dy.$$

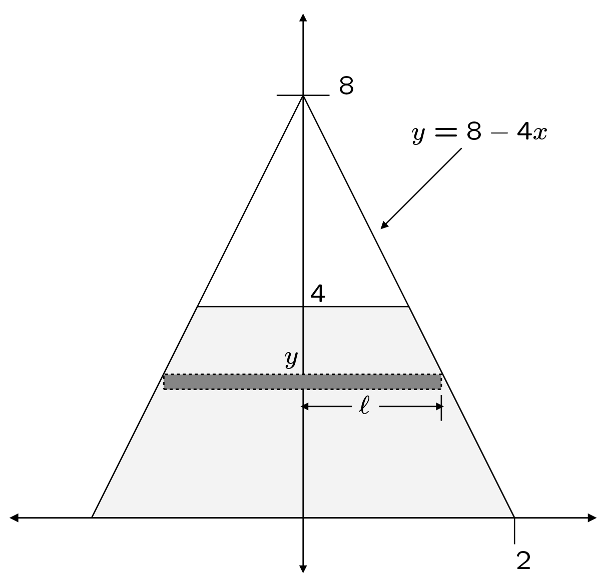
To get the total work for pumping out the tank, we add the last quantity up over all the slabs in the tank. Thus,

$$\begin{aligned}
\text{Total Work} &= \int_0^1 \pi w(2-y)(1-y^2)\,dy \\
&= \pi w \int_0^1 (2-y)(1-y^2)\,dy \\
&= \pi w \int_0^1 (2-2y^2-y+y^3)\,dy \\
&= \pi w \left[2y - \frac{2}{3}y^3 - \frac{y^2}{2} + \frac{y^4}{4}\right]_0^1 \\
&= \frac{13\pi w}{12} \\
&= \frac{13\pi(9800)}{12} \\
&\approx \boxed{33,353.24 \text{ Joule}}\,.
\end{aligned}$$

Problem 7.

The vertical cross sections of a tank are isosceles triangles, point upwards. The bottom of the tank is 4 feet across and it is 8 feet high. If the tank is filled with water to a depth of 4 feet, what is the total force on one end of the tank? (The weight density of water is $w = 62.4\ ^{\text{lbs}}/_{\text{ft}^3}$.)

Answer:
Put in the coordinate system by letting the y-axis run upwards from the center of the base of the tank. Here's the picture:



The light gray area is the part occupied by the water. Take a strip at height y and thickness dy on the end of the tank, shown as the dark gray rectangle. Half the length of the strip is the distance label1ed ℓ. Next, find the equation of the line along the right side of the triangle. This is goes through the points $(2, 0)$ and $((0, 8))$, so it's easy to calculate that the equation is $y = 8 - 4x$. If we solve this for x in terms of y, we get $x = 2 - y/4$. Thus, $\ell = 2 - y/4$. From this we conclude that the length of the strip is $2(2 - y/4) = 4 - y/2$. Since the thickness of the strip is dy, we get

$$\text{Area of strip} = (4 - y/2)\,dy.$$

The top of the water is at $y = 4$, so we have

$$\text{Depth of strip} = 4 - y.$$

The pressure at this depth is

$$\text{Pressure on strip} = w(4 - y).$$

The force on the strip is the pressure times the area, so we have

$$\text{Force on strip} = w(4 - y)(4 - y/2)\,dy.$$

Now we have to add up on the force on all the little strips from $y = 0$ to $y = 4$. So, we can calculate

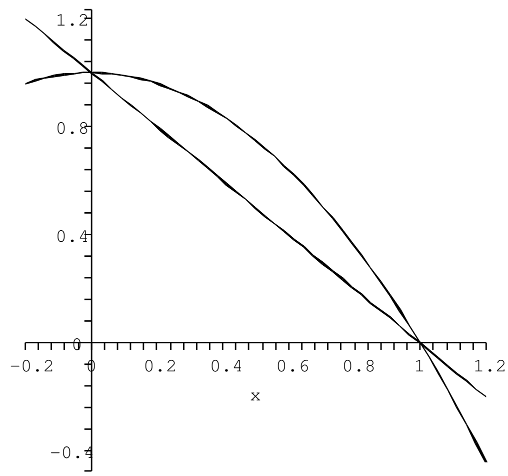
$$\begin{aligned}
\text{Total force} &= \int_0^4 w(4 - y)(4 - y/2)\,dy \\
&= w \int_0^4 [16 - 2y - 4y + y^2/2]\,dy \\
&= w \int_0^4 [16 - 6y + y^2/2]\,dy \\
&= w \left[16y - 3y^2 + \frac{1}{6}y^3 \right]_0^4 \\
&= w \left[16(4) - 3(4)^2 + \frac{4^3}{6} \right] \\
&= w \frac{80}{3} \\
&= (62.4)\frac{80}{3} \\
&= 1664\,\text{lbs.}
\end{aligned}$$

Problem 8. Let R be the region bounded by the curves $y = 1 - x^2$ and $y = 1 - x$.

A. Find the area of R.

Answer:

The picture looks like this:



The points of intersection of the two curves are $(0, 1)$ and $(1, 0)$.

The top curve is the parabola $y = 1 - x^2$ and the bottom curve is the line $y = 1 - x$, so

$$\begin{aligned} A &= \int_0^1 [(1 - x^2) - (1 - x)]\, dx \\ &= \int_0^1 [1 - x^2 - 1 + x]\, dx \\ &= \int_0^1 (x - x^2)\, dx \\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6}. \end{aligned}$$

B. Find the y-coordinate of the centroid of R.

Answer:

The y-coordinate of the centroid $\bar{y}$ is given by $\bar{y} = I_x/A$, where A is the area and I_x, moment with respect to the x-axis is, in general, given by

$$I_x = \frac{1}{2}\int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx.$$

In our case the top curve is $f(x) = 1 - x^2$ and the bottom curve is $g(x) = 1 - x$. Thus,

$$\begin{aligned} I_x &= \frac{1}{2}\int_0^1 \left\{ [1 - x^2]^2 - [1 - x]^2 \right\} dx \\ &= \frac{1}{2}\int_0^1 \{1 - 2x^2 + x^4 - [1 - 2x + x^2]\} \, dx \\ &= \frac{1}{2}\int_0^1 [1 - 2x^2 + x^4 - 1 + 2x - x^2] \, dx \\ &= \frac{1}{2}\int_0^1 [2x - 3x^2 + x^4] \, dx \\ &= \frac{1}{2}\left[x^2 - x^3 + \frac{1}{5}x^5 \right]_0^1 \\ &= \frac{1}{2}\left[\frac{1}{5} \right] \\ &= \frac{1}{10}. \end{aligned}$$

From this we get

$$\bar{y} = \frac{I_x}{A} = \frac{\frac{1}{10}}{\frac{1}{6}} = \frac{3}{5}$$

C. Use the Theorem of Pappus to find the volume of the solid generated when the region R is revolved around the x-axis.

Answer:

Pappus' Theorem says that the volume of the solid of revolution is $V = As$ where A is the area of the region and s is the distance traveled by the centroid. In our case, the centroid travels around a circle of radius $\bar{y}$, so we have

$$s = 2\pi\bar{y} = 2\pi\frac{3}{5} = \frac{6\pi}{5}.$$

Thus, the volume is

$$V = As = \frac{1}{6}\frac{6\pi}{5} = \frac{\pi}{5}.$$

Problem 9.
Find the following integrals.

A.

$$\int x\cos(2x)\,dx.$$

Answer:

Use the integration by parts formula

$$(*) \qquad \boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Set

$$u = x, \qquad v' = \cos(2x),$$

so we have

$$u' = 1, \qquad v = \int \cos 2x\,dx = \frac{1}{2}\sin(2x).$$

Plugging this into $(*)$, we get

$$\begin{aligned}\int x\cos(2x)\,dx &= \frac{1}{2}x\sin(2x) - \int (1)\frac{1}{2}\sin(2x)\,dx \\ &= \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx \\ &= \frac{1}{2}x\sin(2x) - \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right] + C \\ &= \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C.\end{aligned}$$

B.

$$\int x^2\ln(x)\,dx.$$

Answer:

Use the integration by parts formula $(*)$ with

$$u = \ln(x), \qquad v' = x^2,$$

then

$$u' = \frac{1}{x}, \qquad v = \int x^2\,dx = \frac{1}{3}x^3.$$

Plugging into(∗) gives

$$\begin{aligned}\int x^2 \ln(x)\,dx &= \frac{1}{3}x\ln(x) - \int \left[\frac{1}{x}\right]\frac{1}{3}x^3\,dx \\ &= \frac{1}{3}x\ln(x) - \frac{1}{3}\int x^2\,dx \\ &= \frac{1}{3}x\ln(x) - \frac{1}{3}\left[\frac{1}{3}x^3\right] + C \\ &= \frac{1}{3}x\ln(x) - \frac{1}{9}x^3 + C.\end{aligned}$$
