# NOTES ON STABILITY ANALYSIS

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## 1. Introduction

These notes are a direct continuation of the notes "Notes on Linear Systems of Differential Equations." In these notes, we will study the solutions of a linear system of differential equations with constant coefficients,

$$\frac{dy}{dt} = Ay.$$

We want to study the long time behavior of the solutions, in other words, what happens to $y(t)$ as $t \to \infty$.

We begin with some preliminary material.

### 1.1. Norms on Vector Spaces.

We want to measure the "size" or "length" of a vector. The mathematical device for doing this in called a norm. Here is the definition.

**Definition 1.1.** Let $V$ be a vector space over $\mathbb{K}$.[1] A **norm** on $V$ is a function $\|\cdot\|\colon V \to \mathbb{R}\colon v \mapsto \|v\|$ that has the following properties.

(1) $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.
(2) If $\lambda \in \mathbb{K}$ and $v \in V$, $\|\lambda v\| = |\lambda|\,\|v\|$.
(3) For all $v_1, v_2 \in V$, $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$. This is called the **triangle inequality**.

**Exercise 1.2.** Show that $\|-v\| = \|v\|$ and $\|v_1 - v_2\| = \|v_2 - v_1\|$.

The following is a simple consequence of the triangle inequality, which we will include under that name.

**Proposition 1.3.** *For all* $v_1, v_2 \in V$,

$$\big|\, \|v_1\| - \|v_2\| \,\big| \leq \|v_1 \pm v_2\|.$$

*Proof.* We have

$$\begin{aligned} \|v_1\| &= \|v_1 - v_2 + v_2\| \\ &\leq \|v_1 - v_2\| + \|v_2\|, \end{aligned}$$

by the triangle inequality, so

$$\|v_1\| - \|v_2\| \leq \|v_1 - v_2\|.$$

Switching the roles of $v_1$ and $v_2$ gives

$$\|v_2\| - \|v_1\| \leq \|v_2 - v_1\| = \|v_1 - v_2\|.$$

Thus, we have

$$\pm\big[\|v_1\| - \|v_2\|\big] \leq \|v_1 - v_2\|,$$

[1] As usual, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

and so we have

$$\big|\, \|v_1\| - \|v_2\| \,\big| \leq \|v_1 - v_2\|.$$

Replacing $v_2$ by $-v_2$ yields

$$\big|\, \|v_1\| - \|v_2\| \,\big| \leq \|v_1 + v_2\|.$$

□

Two examples of norms on $\mathbb{R}^n$ are the **Euclidean Norm** $\|\cdot\|_{\mathrm{e}}$ which is defined by

$$\|x\|_{\mathrm{e}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

and the **max norm**, defined by

$$\|x\|_{\max} = \max\{|x_1|, |x_2|, \dots, |x_n|\},$$

where $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$. It takes some work (the Cauchy-Schwartz inequality) to show that the Euclidean norm satisfies the triangle inequality, but we won't include the proof here. Similar norms can be defined on $\mathbb{C}^n$. The definition of the max norm looks the same, but $|\cdot|$ means the modulus function in the complex case. For the Euclidean norm, we have to add modulus signs,

$$\|z\|_{\mathrm{e}} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \qquad z \in \mathbb{C}^n.$$

So which norm should we use on $\mathbb{R}^n$? It turns out that form many problems, it doesn't matter. We make the following definition.

**Definition 1.4.** Let $V$ be a vector space over $\mathbb{K}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if there are constants $C_1 \geq 0$ and $C_2 \geq 0$ such that $\|v\|_2 \leq C_1\|v\|_1$ and $\|v\|_1 \leq C_2\|v\|_2$ for all $v \in V$.

It is easy to show that the Euclidean and max norms on $\mathbb{K}^n$ are equivalent. First, note that if $x \in \mathbb{K}^n$, then for any $j$,

$$|x_j|^2 \leq |x_1|^2 + \cdots + |x_j|^2 + \cdots + |x_n|^2,$$

so taking square roots gives

$$|x_j| \leq \|x\|_{\mathrm{e}}.$$

Since this holds for all $j$, so we have

$$\|x\|_{\max} \leq \|x\|_{\mathrm{e}}.$$

To get an inequality in the other direction, note that

$$|x_j| \leq \|x\|_{\max},$$

and so

$$|x_1|^2 + \cdots + |x_n|^2 \leq \underbrace{\|x\|_{\max}^2 + \cdots + \|x\|_{\max}^2}_{n \text{ terms}} = n\|x\|_{\max}^2.$$

Taking square roots gives

$$\|x\|_{\mathrm{e}} \leq \sqrt{n}\|x\|_{\max}.$$

**Exercise 1.5.** Consider the function $\|\cdot\|_{\mathrm{sum}}$ on $\mathbb{K}^n$ defined by

$$\|x\|_{\mathrm{sum}} = |x_1| + |x_2| + \cdots + |x_n|.$$

Show that $\|\cdot\|_{\mathrm{sum}}$ is a norm which is equivalent to the Euclidean norm and the max norm.

The following important theorem is a consequence of the Heine-Borel Theorem. Since we haven't got time to go into that circle of ideas, the proof is omitted.

**Theorem 1.6.** *Let $V$ be a finite dimensional vector space over $\mathbb{K}$, then there are norms on $V$, and all norms on $V$ are equivalent.*

Note that this theorem does *not* hold for infinite dimensional vector spaces.

As an example of the idea of equivalent norms, and for it's own sake, consider the idea of differentiating a function $f\colon \mathbb{R} \to V$, where $V$ is a finite dimensional vector space over $\mathbb{K}$.

Choose a norm $\|\cdot\|_1$ on $V$. As in calculus, we would say that $f$ is differentiable at $t_0$ with derivative $f'(t_0) = v_0$ if

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0} = v_0.$$

Using $\|\cdot\|_1$ as the measure of size, the limit means

$$\lim_{t\to t_0} \left\| \frac{f(t) - f(t_0)}{t - t_0} - v_0 \right\|_1 = 0.$$

If we choose another norm $\|\cdot\|_2$, do we get a *different* notion of derivative? No, because the two norms are equivalent. If $f$ is differentiable with respect to $\|\cdot\|_1$, we have

$$\left\| \frac{f(t) - f(t_0)}{t - t_0} - v_0 \right\|_2 \le C_1 \left\| \frac{f(t) - f(t_0)}{t - t_0} - v_0 \right\|_1 \to 0$$

for some constant $C_1$, and so

$$\lim_{t\to t_0} \left\| \frac{f(t) - f(t_0)}{t - t_0} - v_0 \right\|_2 = 0.$$

Thus, $f$ is differentiable (with the same derivative) with respect to $\|\cdot\|_2$. Similarly, if $f$ is differentiable with respect of $\|\cdot\|_2$, then it is differentiable with respect to $\|\cdot\|_1$. Thus, all norms on $V$ give us the same notion of differentiation.

Suppose now that $v_1, \dots, v_n$ is a basis of $V$. Let $\|\cdot\|$ be the max norm with respect to this basis, i.e., if $v = c_1 v_1 + \cdots + c_n v_n$. then

$$\|v\| = \max\{|c_1|, \dots, |c_n|\}.$$

Another way to say this is

$$\|v\| = \| \, [v]_{\mathcal{V}} \, \|_{\max}$$

where $\mathcal{V} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ and $\|\cdot\|_{\max}$ is the max norm on $\mathbb{K}^n$.

We can write

$$f(t) = \sum_{j=1}^{n} f_j(t) v_j$$

for scalar-valued functions $f_j(t)$. Suppose that $f$ is differentiable at $t_0$ and $f'(t_0) = w$, where

$$w = \sum_{j=1}^{n} c_j v_j \tag{1.1}$$

We have

$$\frac{f(t) - f(t_0)}{t - t_0} - w = \sum_{j=1}^{n} \left[ \frac{f_j(t) - f_j(t_0)}{t - t_0} - c_j \right] v_j,$$

so then

$$\left| \frac{f_j(t) - f_j(t_0)}{t - t_0} - c_j \right| \leq \left\| \frac{f(t) - f(t_0)}{t - t_0} - w \right\| \to 0,$$

and we conclude that each $f_j$ is differentiable at $t_0$ with $f_j'(t_0) = c_j$.

Conversely, suppose that each $f_j$ is differentiable at $t_0$ with derivative $f_j'(t_0) = c_j$. Let $w$ be the vector defined by (1.1). We then have

$$\left| \frac{f_j(t) - f_j(t_0)}{t - t_0} - c_j \right| \to 0$$

for each $j$, and so

$$\left\| \frac{f(t) - f(t_0)}{t - t_0} - w \right\| \to 0$$

Thus, $f$ is differentiable at $t_0$ with $f'(t_0) = w$.

We conclude that to differentiate a function with values in $V$, you can just differentiate its components with respect to a basis, i.e.,

$$f'(t) = \sum_{j=1}^{n} f_j'(t) v_j$$

and that this works for *every* basis.

1.2. **Norms on Linear Transformations.** Suppose that $V$ and $W$ are finite dimensional vector spaces over $\mathbb{K}$. Let

$$L(V, W) = \{L \colon V \to W \mid L \text{ linear}\}$$

be the space of linear transformations from $V$ to $W$. The space $L(V, W)$ is a vector space over $\mathbb{K}$ with respect to the usual operations

$$(L_1 + L_2)(v) = L_1(v) + L_2(v)$$
$$(\lambda L)(v) = \lambda(L(v)).$$

This vector space is finite dimensional, since choosing bases for $V$ and $W$ puts $L(V, W)$ in a one-to-one linear correspondence with a space of matrices.

Since $L(V, W)$ is finite dimensional, all norms on this space are equivalent. However, it is useful to have a relationship between the norm you are using on $L(V, W)$ and the norms you are using on $V$ and $W$.

To construct this norm on $L(V, W)$, suppose that we have fixed norms on $V$ and $W$, both denoted by $\|\cdot\|$ (it should be clear from context which space you're in). We have the following Lemma.

**Lemma 1.7.** *If $L \in L(V, W)$ there is a constant $C$ so that $\|L(v)\| \leq C\|v\|$ for all $v \in V$.*

*Proof.* Choose a basis $\mathcal{V} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ for $V$ and let $\|\cdot\|_{\mathcal{V}}$ be the max norm with respect to this basis, i.e,

$$\|v\|_{\mathcal{V}} = \left\| [v]_{\mathcal{V}} \right\|_{\max}.$$

Since all norms on $V$ are equivalent, there is a constant $K$ so that $\|v\|_{\mathcal{V}} \leq K\|v\|$ for all $v \in V$.

Now, suppose that

$$v = \sum_{j=1}^{n} c_j v_j.$$

Then, we have

$$\begin{aligned}
\|L(v)\| &= \left\|\sum_{j=1}^{n} c_j L(v_j)\right\| \\
&\leq \sum_{j=1}^{n} |c_j|\,\|L(v_j)\| && \text{(triangle inequality)} \\
&\leq \sum_{j=1}^{n} \|v\|_{\mathcal{V}} \|L(v_j)\| \\
&\leq \sum_{j=1}^{n} K\|v\|\,\|L(v_j)\| \\
&= \left[K \sum_{j=1}^{n} \|L(v_j)\|\right] \|v\|,
\end{aligned}$$

so we can take $C$ to be

$$K \sum_{j=1}^{n} \|L(v_j)\|.$$

□

We now define the **operator norm** on $L(V, W)$ corresponding to our chosen norms on $V$ and $W$ as

$$\|L\| = \max\{\|L(v)\| \mid \|v\| = 1\} = \max\left\{\frac{\|L(v)\|}{\|v\|} \mid v \neq 0\right\}. \tag{1.2}$$

First note that the two sets of numbers in this equation are the same. Certainly,

$$\{\|L(v)\| \mid \{\|L(v)\| \mid \|v\| = 1\}\|v\| = 1\} \subseteq \left\{\frac{\|L(v)\|}{\|v\|} \mid v \neq 0\right\}$$

since $\|L(v)\| = \|L(v)\|/\|v\|$ when $\|v\| = 1$. To see that

$$\left\{\frac{\|L(v)\|}{\|v\|} \mid v \neq 0\right\} \subseteq \{\|L(v)\| \mid \|v\| = 1\},$$

note that

$$\frac{\|L(v)\|}{\|v\|} = \left\|L\left(\frac{v}{\|v\|}\right)\right\|$$

and that $v/\|v\|$ has norm 1.

This set of numbers is bounded above by the last lemma. The maximums in (1.2) exist because of the Heine-Borel Theorem.

Of course, to justify the notation $\|L\|$, we have to show that (1.2) satisfies the definition of a norm. Before undertaking this, we prove a lemma which we will call "the basic lemma" for operator norms.

**Lemma 1.8.** *If $L\colon V \to W$ is linear, then*

$$\|L(v)\| \leq C\|v\|, \qquad \text{for all } v \in V, \tag{1.3}$$

*if and only if $C \geq \|L\|$. In particular, we have*

$$\|L(v)\| \leq \|L\|\,\|v\|, \qquad \text{for all } v \in V.$$

*Proof.* Suppose that (1.3) holds. Then, if $\|v\| = 1$, we have $\|L(v)\| \leq C$. Since this holds for all vectors $v$ on norm 1, we have $\|L\| \leq C$.

Conversely, suppose that $C \geq \|L\|$. Then, for any vector $w \neq 0$, we have

$$C \geq \|L\| = \max\left\{ \frac{\|L(v)\|}{\|v\|} \mid v \neq 0 \right\} \geq \frac{\|L(w)\|}{\|w\|}.$$

Thus, $\|L(w)\| \leq C\|w\|$. Since $w$ was arbitrary, we conclude that

$$\|L(v)\| \leq C\|v\|$$

for al1 $v \neq 0$. It is also true for $v = 0$, since both sides are 0 in that case. □

**Proposition 1.9.** *The function* $\|\cdot\|$ *defined by* (1.2) *is a norm on* $L(V, W)$.

*Proof.* We have $\|L\| \geq 0$, since $\|L\|$ is the max of a set on non-negative numbers. If $L$ is the zero transformation, then the sets of numbers in (1.2) are $\{0\}$, so $\|0\| = 0$. On the other hand, if $\|L\| = 0$, then

$$\|L(v)\| \leq \|L\| \, \|v\| = 0,$$

so $L(v) = 0$ for all $v$. Thus, $L$ is the zero transformation.

Let $\lambda$ be a scalar. If $\lambda = 0$, clearly $\|\lambda L\| = |\lambda| \, \|L\|$. If $\lambda \neq 0$,

$$\begin{aligned} \|\lambda L\| &= \max\{\|((\lambda L)(v)\| \mid \|v\| = 1\} \\ &= \max\{|\lambda| \, \|L(v)\| \mid \|v\| = 1\}. \end{aligned}$$

The function $m \colon \mathbb{R} \to \mathbb{R} \colon x \to |\lambda| x$ is one-to-one and order preserving, so

$$\max\{|\lambda| \, \|L(v)\| \mid \|v\| = 1\}. = |\lambda| \max\{\|L(v)\| \mid \|v\| = 1\} = |\lambda| \, \|L\|.$$

Thus, we have $\|\lambda L\| = |\lambda| \, \|L\|$.

Finally, to prove the triangle inequality, note that for all $v \in V$,

$$\begin{aligned} \|(L_1 + L_2)(v)\| &= \|L_1(v) + L_2(v)\| \\ &\leq \|L_1(v)\| + \|L_2(v)\| \\ &\leq \|L_1\| \, \|v\| + \|L_2\| \, \|v\| \\ &= \big[\|L_1\| + \|L_2\|\big] \, \|v\|. \end{aligned}$$

By our basic lemma, this implies

$$\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|.$$

This completes the proof. □

The next proposition gives another important property of the operator norms.

**Proposition 1.10.** *Let* $U$, $V$ *and* $W$ *be vector spaces over* $\mathbb{K}$ *and let* $L_1 \colon U \to V$ *and* $L_2 \colon V \to W$ *be linear transformations. Choose norms on* $U$, $V$ *and* $W$ *and use the corresponding operator norms on* $L(U, V)$, $L(V, W)$ *and* $L(U, W)$. *Then,*

$$\|L_2 L_1\| \leq \|L_2\| \, \|L_1\|.$$

*Proof.* For all $u \in U$, we have

$$\|(L_2 L_1)(v)\| = \|L_2(L_1(v))\| \leq \|L_2\| \, \|L_1(v)\| \leq \|L_2\| \, \|L_1\| \, \|v\|.$$

But then the basic lemma tells us $\|L_2 L_1\| \leq \|L_2\| \, \|L_1\|$. □

In these notes, we will almost always us the max norm on $\mathbb{K}^n$, which we will denote by just $\|\cdot\|$, if no confusion will result. Similarly, we will almost always use the operator norm induced by the max norms as the norm on $L(\mathbb{K}^n, \mathbb{K}^m)$ and we will denote it by just $\|\cdot\|$. Of course, an $m \times n$ matrix $A$ induces a linear transformation $\mathbb{K}^n \to \mathbb{K}^m$ by left multiplication, and we will just write $\|A\|$ for the operator norm of this transformation. Since we're using the max norms, the operator norm is actually easy to compute.

**Proposition 1.11.** *Let $A$ be an $m \times n$ matrix. If we equip $\mathbb{K}^n$ and $\mathbb{K}^m$ with the max norms, the corresponding operator norm is given by*

$$\|A\| = \max\left\{ \sum_{k=1}^{n} |a_{jk}| \mid j = 1, \ldots, n \right\}, \tag{1.4}$$

*in other words, for each row in the matrix, form the sum of the absolute values of the entries in the row. The operator norm is the largest of these sums.*

*Proof.* Let $M$ be the number defined by the right-hand side of (1.4). Suppose $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ and let $y = Ax$. For any $j$, we have

$$\begin{aligned} |y_j| &= \left| \sum_{k=1}^{n} a_{jk} x_k \right| \\ &\leq \sum_{k=1}^{n} |a_{jk}||x_k| \\ &\leq \sum_{k=1}^{n} |a_{jk}| \, \|x\| \\ &= \left[ \sum_{k=1}^{n} |a_{jk}| \right] \|x\| \\ &\leq M\|x\|. \end{aligned}$$

Thus, we have $|y_j| \leq M\|x\|$. Since $\|y\|$ is the maximum of the $|y_j|$'s, we have $\|y\| \leq M\|x\|$. Our basic lemma then gives us

$$\|A\| \leq M.$$

It remains to prove the converse inequality.

Recall that is $z$ is a complex number, there is a complex number $\operatorname{csgn}(z)$ such that $|\operatorname{csgn}(z)| = 1$ and $\operatorname{csgn}(z)z = |z|$. Indeed, if $z \neq 0$, $\operatorname{csgn}(z)$ must be $|z|/z$ and we define $\operatorname{csgn}(0) = 1$. The name csgn stands for complex sign, since, for a nonzero real number $x$, $\operatorname{csgn}(x) = 1$ if $x > 0$ and $\operatorname{csgn}(x) = -1$ if $x < 0$.

Now choose $\ell$ so that

$$\sum_{k=1}^{n} |a_{\ell k}| = M.$$

Let $v$ be the vector

$$v = \begin{bmatrix} \operatorname{csgn}(a_{\ell 1}) & \operatorname{csgn}(a_{\ell 2}) & \ldots & \operatorname{csgn}(a_{\ell n}) \end{bmatrix}^T.$$

Let $y = Av$. We then have

$$\begin{aligned} y_\ell &= \sum_{k=1}^n a_{\ell k} v_k \\ &= \sum_{k=1}^n a_{\ell k} \operatorname{csgn}(a_{\ell k}) \\ &= \sum_{k=1}^n |a_{\ell k}| \\ &= M \end{aligned}$$

Note that $\|v\| = 1$, so we have

$$\begin{aligned} \|A\| &= \max\{\|Ax\| \mid \|x\| = 1\} \\ &\geq \|Av\| \\ &= \|y\| \\ &\geq |y_\ell| \\ &= M. \end{aligned}$$

Thus, we have $M \leq \|A\|$ and the proof is complete. □

*Remark* 1.12. If we use the Euclidean norms on the spaces $\mathbb{K}^n$, the corresponding operator norms are harder to compute. We'll give the result without proof.

If $A$ is a $m \times n$ matrix, the operator norm of $A$ with respect to the Euclidean norms is the square root of the largest eigenvalue of $A^T A$. Note that $A^T A$ is square and that it can be shown the eigenvalues of $A^T A$ are non-negative.

In the case where $A$ is square and diagonalizable, the number above is the same as the largest absolute value of an eigenvalue of $A$.

## 2. Stability Analysis

We want to study what happens to the solutions of a linear system

$$\frac{dy}{dt} = Ay$$

as $t \to \infty$. Here $A$ is an $n \times n$ real or complex matrix.

**Exercise 2.1.** If $y_0$ is an eigenvalue of $A$ with eigenvalue $\lambda$, show that the solution of the initial value problem

$$\frac{dy}{dt} = Ay, \quad y(0) = y_0$$

is

$$y(t) = e^{\lambda t} y_0.$$

If $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, then

$$e^{\lambda t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

and so

$$|e^{\lambda t}| = e^{\alpha t}.$$

To see what the other part contributes, we want to view things from a real perspective. Let $S$ be the one-dimensional complex subspace of $\mathbb{C}^n$ Any vector

$s \in S$ can be written as $\zeta v$. Let $\zeta = a + ib$, where $a$ and $b$ are real numbers. Then $\zeta v = av + b(iv)$. Thus, we can consider $S$ as a two dimensional real vector space with basis $v$ and $iv$ (why are they independent?). Let $\Phi(t)\colon S \to S$ be the linear transformation given by multiplication by $e^{i\beta t}$. We want to find the matrix of this transformation with respect to the ordered basis $\begin{bmatrix} v & iv \end{bmatrix}$.

We have

$$\begin{aligned} e^{i\beta t} v &= (\cos(\beta t) + i\sin(\beta t))v = \cos(\beta t)v + \sin(\beta t)(iv) \\ e^{i\beta t} iv &= (\cos(\beta t) + i\sin(\beta t))iv = -\sin(\beta t)v + \cos(\beta t)(iv) \end{aligned}$$

and hence

$$\begin{bmatrix} \Phi(t)(v) & \Phi(t)(iv) \end{bmatrix} = \begin{bmatrix} v & iv \end{bmatrix} \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}.$$

Hence the matrix of $\Phi(t)$ with respect to this basis is

$$\begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix}.$$

A little thought shows this is the matrix of counter-clockwise rotation through $\beta t$ radians. Thus, $e^{i\beta t}v$ travels around a circle in $S$, from the real point of view.

Now consider

$$e^{\lambda t} v = e^{\alpha t}\Phi(t)v$$

If $\alpha < 0$, then $e^{\alpha t} \to 0$ as $t \to \infty$. Thus, $e^{\lambda t}v$ spirals into the origin as $t \to \infty$.

If $\alpha > 0$, $e^{\lambda t}v$ spirals outward toward infinity.

Of course, if $\alpha = 0$, we've seen that $e^{\lambda t}v$ travels around a circle and we have a periodic orbit.

**Exercise 2.2.** What happens if $\beta = 0$?

Of course, in general, we have to deal with generalized eigenvectors. We will first consider the cases where all of the eigenvalues have negative real part, positive real part, an zero real part, and then combine these cases.

2.1. **All Eigenvalues have Negative Real Part.** We begin with recalling some facts from calculus. Recall that if $k > 0$ and $p$ is any non-negative integer,

$$\lim_{t \to 0} t^p e^{-kt} = 0$$

(this follows from L'Hôpital's rule). Since $t^p e^{-kt}$ is continuous on $[0, \infty)$, we conclude that $t^p e^{-kt}$ is bounded on $[0, \infty)$. In other words, there is some constant $K$ such that

$$t^p e^{-kt} \le K, \qquad t \ge 0.$$

If we have a polynomial $p(t)$ with complex coefficients, say

$$p(t) = \sum_{j=0}^{m} c_j t^j,$$

then $|p(t)e^{-kt}|$ is bounded, since

$$\begin{aligned} |p(t)e^{-kt}| &= e^{-kt}\left|\sum_{j=0}^{m} c_j t^j\right| \\ &\le \sum_{j=0}^{\infty} |c_j| t^j e^{-kt} \\ &\le \sum_{j=0}^{\infty} |c_j| K_j \\ &= \text{Constant}, \end{aligned}$$

where $t^j e^{-kt} \le K_j$.

Thus, we conclude that for any polynomial,

$$|p(t)e^{-kt}| \le K, \qquad t > 0.$$

Now consider the case of an $n \times n$ matrix $A$ with eigenvalues $\lambda_1, \dots, \lambda_r$, where all the eigenvalues have negative real parts. We can find some number $\sigma > 0$ such that

$$\operatorname{Re}(\lambda_j) < -\sigma, \qquad j = 1, \dots, r.$$

Let $p(t)$ be a polynomial, and consider

$$e^{\sigma t} p(t) e^{\lambda_j t} = p(t) e^{(\sigma + \lambda_j)t}$$

Since $\sigma + \lambda_j < 0$, there is a constant $K$ so that

$$|e^{\sigma t} p(t) e^{\lambda_j t}| \le K, \qquad t \ge 0.$$

But then we have

$$|p(t)e^{\lambda_j t}| \le K e^{-\sigma t}, \qquad t \ge 0.$$

Thus, $p(t)e^{\lambda_j t}$ goes to zero exponentially fast.

With this preparation, we can prove the following proposition.

**Proposition 2.3.** *Let $A$ be an $n \times n$ matrix and assume that all of the eigenvalues of $A$ have negative real part. Then there is a number $\sigma > 0$ and a constant $K$ such that*

$$\|e^{At}\| \le K e^{-\sigma t}, \qquad t \ge 0.$$

*Proof.* Denote the entries of $e^{At}$ by $\varphi_{ij}(t)$. We know each $\varphi_{ij}(t)$ is a sum of functions of the form $p(t)e^{\lambda_j t}$, where $\lambda_j$ is an eigenvalue of $A$. Thus, we have $abs \varphi_{ij}(t) \le K e^{-\sigma t}$ for some constant $K$.

We'll use $K$ to stand for a generic constant, which may be different in different appearances.

For any row, we have

$$\sum_{k=1}^{n} |\varphi_{jk}| \le \sum_{k=1}^{n} K e^{-\sigma t} \le K e^{-\sigma t}.$$

Thus,

$$\|e^{At}\| = \max_j \sum_{k=1}^{n} |\varphi_{jk}(t)| \le K e^{-\sigma t}.$$

□

We can apply this to the trajectories of the system

$$\frac{dy}{dt} = Ay, \tag{2.1}$$

where all the eigenvalues of $A$ have negative real part, as follows. Note that the solution to this system with initial condition $y_0 = 0$ is $y(t) = 0$ for all $t$. Thus, if we start the system at the origin, it just stays there. We say the origin is an **equilibrium point** or a **fixed point** of the system. (The terms **critical point** and **singular point** are also used.). For any other initial condition $y_0$, the solution is $y(t) = e^{At}y_0$. We then have

$$\begin{aligned} \|y(t)\| &= \|e^{At}y_0\| \\ &\leq \|e^{At}\|\,\|y_0\| \\ &\leq K\|y_0\|e^{-\sigma t} \to 0 \end{aligned}$$

Thus, all other solutions approach the origin as $t \to \infty$.

In this case we say that the fixed point at the origin is **globally asymptotically stable**. The idea is this. Suppose that (2.1) represents a mechanical system that we want to stay in equilibrium at the origin. If some error or disturbance causes the system to go to state $y_0 \neq 0$, things aren't so bad, because the disturbance dies out and the system evolves back toward the desired equilibrium state.

2.2. **All Eigenvalues have Positive Real Part.** Next consider the case where all of the eigenvalues of $A$ have (strictly) positive real part. In this case we can find a number $\sigma$ such that

$$0 < \sigma < \mathrm{Re}(\lambda_j) \tag{2.2}$$

for all eigenvalues $\lambda_j$ of $A$.

Suppose the eigenvalues of $A$ are $\lambda_1, \lambda_2, \dots, \lambda_r$. It is easy to see that the eigenvalues of $-A$ are $-\lambda_1, -\lambda_2, \dots, -\lambda_j$. Indeed, if $\lambda$ is an eigenvalue of $A$ with eigenvector $v$, we have $Av = \lambda v$. Taking the negative of both sides, we have $-Av = -\lambda v$, which can be thought of as $(-A)v = (-\lambda)v$. Thus, $-\lambda$ is an eigenvalue of $-A$. Conversely, if $(-A)v = \mu v$, then $Av = (-\mu)v$, so $-\mu$ is an eigenvalue of $A$, and so $\mu$ is the negative of an eigenvalue of $A$.

From (2.2) we have $-\lambda_j < -\sigma < 0$, so our previous work tells us that

$$\|e^{-At}\| = \|e^{(-A)t}\| \leq Ke^{-\sigma t}, \qquad t \geq 0.$$

We then have

$$\begin{aligned} 1 &= \|I\| \\ &= \|e^{-At}e^{At}\| \qquad\qquad \leq \|e^{-At}\|\,\|e^{At}\| \\ &\leq Ke^{-\sigma t}\|e^{At}\|. \end{aligned}$$

Thus, with a different constant $K$ we have

$$K^{e^{\sigma t}} \leq \|e^{At}\|,$$

so we conclude that

$$\|e^{At}\| \to \infty$$

as $t \to \infty$.

We can say more. Consider the trajectories of

$$\frac{dy}{dt} = Ay. \tag{2.3}$$

Of course, the origin is a fixed point. For any other initial condition $y_0$, the trajectory is $y(t) = e^{At}y_0$. We have

$$\begin{aligned}\|y_0\| &= \|e^{-At}e^{At}y_0\| \\ &\leq \|e^{-At}\|\,\|e^{At}y_0\| \\ &\leq Ke^{-\sigma t}\|e^{At}y_0\|.\end{aligned}$$

Thus, for a different constant $K$,

$$\|y(t)\| = \|e^{At}y_0\| \geq K\|y_0\|e^{\sigma t} \to \infty$$

as $t \to \infty$. Thus, all trajectories other that the fixed point at the origin run off to infinity. We would certainly say that the fixed point at the origin is unstable: a small error causes the system to go away from the desired equilibrium.

2.3. **Eigenvalues with Zero Real Part.** . Suppose that $\lambda$ is an eigenvalue of $A$ with zero real part, i.e., $\lambda = i\beta$, $\beta \in \mathbb{R}$.

In this case, consider the generalized eigenspace $G = G(\lambda)$. We have already seen that if $y_0 \in G$ is an eigenvector, $e^{At}y_0 = e^{\lambda t}y_0 = e^{i\beta t}y_0$.

If $\beta = 0$ so $\lambda = 0$, $e^{At}y_0 = y_0$ for all $t$, so $y_0$ is a fixed point. If $\lambda \neq 0$, we have seen that the trajectory is periodic. We have

$$\|e^{At}y_0\| = \|e^{i\beta t}y_0\| = |e^{i\beta t}|\|y_0\| = \|y_0\|,$$

so this trajectory goes around a circle.

What if $y_0$ is a generalized eigenvector, but not an eigenvector? Recall that on $G$, $A = \lambda I + N$ where $N$ is nilpotent. Thus, $Ay_0 = \lambda y_0 + Ny_0$, so $y_0$ is an eigenvector if and only if $Ny_0 = 0$.

If $Ny_0 \neq 0$, we have $e^{At}y_0 = e^{\lambda t}e^{Nt}y_0$ and $\|e^{At}y_0\| = \|e^{tN}y_0\|$, since $|e^{t\lambda}| = 1$. Thus, we need to consider $e^{tN}y_0$. Since $N$ is nilpotent, there is some power $p$ such that $N^p \neq 0$ but $N^{p+1} = 0$. As we know,

$$e^{tN} = I + tN + \frac{t^2}{2!} + \cdots + \frac{t^p}{p!}N^p.$$

Let $q$ be the integer such that $N^q y_0 \neq 0$ but $N^{q+1}y_0 = 0$. Since we're assuming $Ny_0 \neq 0$, we have $1 \leq q \leq p$. Using our expression for $e^{tN}$, we have

$$e^{tN}y_0 = y_0 + tNy_0 + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{q-1}}{(q-1)!} + \frac{t^q}{q!}N^q y_0.$$

We can rewrite this as

$$e^{tN}y_0 = t^q[\frac{1}{q!}N^q y_0 + M(t)],$$

where

$$M(t) = \frac{1}{t^q}y_0 + \frac{1}{t^{q-1}2!}N^2y_0 + \cdots + \frac{1}{t(q-1)!}N^{q-1}y_0.$$

By the triangle inequality

$$\|M(t)\| \leq \frac{1}{t^q}\|y_0\| + \frac{1}{t^{q-1}2!}\|N^2y_0\| + \cdots + \frac{1}{t(q-1)!}\|N^{q-1}y_0\|$$

and so

$$\|M(t)\| \to 0, \qquad \text{as } t \to \infty.$$

Since $\|M(t)\| \to 0$, we will have

$$\|M(t)\| \leq \frac{1}{2q!}\|N^q y_0\|$$

for all sufficiently large $t$. Then, by the triangle inequality, we have

$$\begin{aligned}
\|e^{tN}y_0\| &= \|t^q[\frac{1}{q!}N^q y_0 + M(t)]\| \\
&= t^q\|\frac{1}{q!}N^q y_0 + M(t)\|] \\
&\geq t^q\left[\|\frac{1}{q!}N^q y_0\| - \|M(t)\|\right] \\
&\geq t^q\left[\|\frac{1}{q!}N^q y_0\| - \frac{1}{2q!}\|N^q y_0\|\right] \\
&\geq \frac{t^q}{2q!}\|N^q y_0\|
\end{aligned}$$

for all sufficiently large $t$. Thus, $\|e^{tN}y_0\|$ goes to infinity at a polynomial rate.

To summarize, the eigenvectors in $G$ follow periodic orbits. The vectors in $G$ that are not eigenvectors give rise to trajectories that go off to infinity.

2.4. **Exponentials of Linear Transformations.** In these notes, we have introduced the idea of the derivative of a function $f\colon \mathbb{R} \to V$, where $V$ is a vector space over $\mathbb{K}$. With this notion, we can define the exponential of a matrix as well as the exponential of a matrix.

If $L\colon V \to V$ is a linear transformation, we define $\Phi_L(t) = e^{tL}$ to be the function $\Phi\colon \mathbb{R} \to L(V,V)$ such that solves the initial value problem

$$\frac{d}{dt}\Phi(t) = L\Phi(t), \quad \Phi(0) = I. \tag{2.4}$$

If we choose a basis $\mathcal{V}$ of $V$, then

$$[\Phi'(t)]_{\mathcal{V}\mathcal{V}} = \frac{d}{dt}[\Phi(t)]_{\mathcal{V}\mathcal{V}}$$

(exercise).

Let $A = [L]_{\mathcal{V}\mathcal{V}}$. Since the matrix operations are chosen to reflect the operations on linear transformations, applying $[\cdot]_{\mathcal{V}\mathcal{V}}$ to (2.4) gives

$$\frac{d}{dt}[\Phi(t)]_{\mathcal{V}\mathcal{V}} = [L\Phi(t)]_{\mathcal{V}\mathcal{V}} = [L]_{\mathcal{V}\mathcal{V}}[\Phi(t)]_{\mathcal{V}\mathcal{V}} = A[\Phi(t)]_{\mathcal{V}\mathcal{V}}$$

and, of course,

$$[\Phi(0)]_{\mathcal{V}\mathcal{V}} = [I]_{\mathcal{V}\mathcal{V}} = I.$$

Thus, we conclude that $[\Phi(t)]_{\mathcal{V}\mathcal{V}} = e^{At}$, or, in other words,

$$[e^{Lt}]_{\mathcal{V}\mathcal{V}} = e^{t[L]_{\mathcal{V}\mathcal{V}}}.$$

We can apply this to direct sums as follows. For simplicity, we deal with two summands, but the general case is the same.

**Proposition 2.4.** *Let $V$ be a finite dimensional vector space over $\mathbb{K}$, and suppose that $V = V_1 \oplus V_2$ for two subspaces $V_1$ and $V_2$. Let $L\colon V \to V$ be a linear transformation and suppose that $V_1$ and $V_2$ are invariant under $L$, so we can write $L = L_1 \oplus L_2$, where $L_j\colon V_j \to V_j$, $j = 1, 2$ is the transformation induced by $L$. Then $V_1$ and $V_2$ are invariant under $e^{Lt}$ and $e^{Lt} = e^{L_1 t} \oplus e^{L_1 t}$.*

*Proof.* Let $\Phi(t) = e^{tL_1} \oplus e^{tL_2}$. By definition, this leaves $V_1$ and $V_2$ invariant.

If $v \in V$ and we write $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$, we have

$$\Phi(t)v = e^{tL_1}v_1 + e^{tL_2}v_2.$$

Taking the derivative of this gives

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= L_1 e^{tL_1}v_1 + L_2 e^{tL_2}v_2 \\ &= (L_1 \oplus L_2)(e^{tL_1}v_1 + e^{tL_2}v_2) \\ &= L\Phi(t)v. \end{aligned}$$

We also obviously have $\Phi(0) = I$. Thus, we must have $\Phi(t) = e^{tL}$. □

2.5. **The General Case of Mixed Eigenvalues.** Consider the case of a general $n \times n$ matrix $A$. In this part, we look at $A$ as a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$.

We can put the eigenvalues of $A$ into three classes: The eigenvalues with negative real part, called **stable eigenvalues**; the eigenvalues with zero real part, called **center eigenvalues**; and the eigenvalues with positive real part, called **unstable eigenvalues**.

Let $E_s$ be the direct sum of the generalized eigenspaces corresponding to the stable eigenvalues, let $E_c$ be the direct sum of the generalized eigenspaces corresponding to the center eigenvalues and let $E_u$ be the direct sum of the generalized eigenspaces corresponding to the unstable eigenvalues. Thus, we have

$$\mathbb{C}^n = E_s \oplus E_u \oplus E_c.$$

Each of these spaces is invariant under $A$, so we can write $A = A_s \oplus A_u \oplus A_c$, where $A_s\colon E_s \to E_s$, $A_u\colon E_s \to E_s$ and $A_c\colon E_c \to E_c$ are the induced transformations. Thus, $e^{tA} = e^{tA_s} \oplus e^{tA_u} \oplus e^{tA_c}$.

On $E_s$, $e^{At}$ agrees with $e^{A_s t}$. Since all the eigenvalues of $A_s$ have negative real part, $e^{At}v = e^{A_s t}v \to 0$ as $t \to infinity$, for every $v \in E_s$. Similarly, if $v \in E_u$, then $e^{At}u \to \infty$.

What happens in $E_c$ is a little more complicated. If $v \in E_c$ is an eigenvector, it follows a periodic orbit. More generally, we will have

$$E_c = \bigoplus_{j=1}^{r} G(\mu_j) \tag{2.5}$$

where $\mu_1, \dots, \mu_r$ are the eigenvalues with zero real part. The general element of $E_c$ looks like $v = v_1 + \cdots + v_r$ with $v_j \in G(\mu_j)$.

If all of $v_1, \dots, v_r$ are eigenvectors, then we have

$$e^{At}v = e^{\mu_1 t}v_1 + \cdots + e^{\mu_r t}v_r.$$

Here each component is periodic, but the periods may rationally related, so the sum is in general not periodic. It's usually called quasi-periodic. In any case, this solution is bounded.

If $v \in G(\mu_j)$ and $v$ is not an eigenvector, we know that $e^{At}v \to \infty$. If $v = v_1 + \cdots + v_r$ as above, and one of the $v_j$'s is not an eigenvector, we will have $e^{At}v \to \infty$. To see this, suppose that $e^{At}v_k \to \infty$ and let $P_k \colon E_c \to G(\mu_k)$ be the projection defined by the direct sum decomposition (2.5), i.e., $P(v) = v_k$. Since $P$ is a linear map, there is a constant $C$ so that $\|P(v)\| \leq C\|v\|$. Thus, we have

$$\|e^{At}v_k\| \leq C\|e^{At}v\|.$$

Since the left-hand side goes to infinity, so must the right-hand side.

We can summarize the consequences of this for the stability of the origin, as follows.

**Proposition 2.5.** *Let $A$ be an $n \times n$ matrix, and consider the differential equation*

$$\frac{d(}{dt}y) = Ay$$

*for a function $y \colon \mathbb{R} \to \mathbb{C}^n$. This equation has the origin as a fixed point.*

1. *If there are any unstable eigenvalues, the origin is an unstable fixed point.*
2. *If all the eigenvalues are stable, the origin is asymptotically stable.*
3. *If there are center eigenvalues and all the other eigenvalues are stable, there are two cases. If $A_c$ is diagonalizable on $E_c$, the origin is stable, but not asymptotically stable. If $A_c$ is not diagonalizable, the origin is unstable.*

2.6. **Real Dynamics.** In the case where $A$ is an $n \times n$ real matrix, you might wonder how the above results apply, since the generalized eigenspaces may not contain any real vectors. What is does the dynamics look like on the real vectors?

To examine this, we introduce the idea of conjugacy. Suppose that $V$ and $W$ are vector spaces over $\mathbb{K}$ and that $L_1 \colon V \to V$ and $L_2 \colon W \to W$ are linear transformations.

Suppose that $T \colon V \to W$ is one-to-one linear correspondence between $V$ and $W$ so that the diagram

$$\begin{array}{ccc} V & \xrightarrow{L_1} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[L_2]{} & W \end{array}$$

commutes, i.e., $TL_1 = L_2T$. In this case, we say $L_1$ and $L_2$ are conjugate by $T$. The idea is that $T$ provides a one-to-one correspondence between $V$ and $W$ and $L_1$ and $L_2$ correspond. In other words, $L_2 = TL_1T^{-1}$. Thus, to find the action of $L_2$ on $w \in W$, you can first find the point $v = T^{-1}(w)$ that corresponds to $w$ in $V$, apply $L_1$ to this corresponding point to get $L_1v = L_1T^{-1}v$, and then take the point $TL_1v = TL_2T^{-1}w$ that corresponds to $L_1v$ in $W$. Thus, $L_1$ and $L_2$ are the same under this correspondence.

We can carry this further. We must have $Te^{tL_1} = e^{tL_2}T$ for all $t \in \mathbb{R}$, or $Te^{tL_1}T^{-1} = e^{tL_2}$. So, on one hand, you can take the trajectory $e^{tL_2}w$ starting at an elemet $w \in W$, or you can go to the corresponding point $v = T^{-1}w$, following the trajectory $e^{tL_1}v$ starting at that point, and then take the corresponding point $Te^{tL_1}v$ in $W$. The result is the same. Thus $T$ sends the trajectories of $L_1$ in $V$ to the trajectories of $L_2$ in $W$. We can say that the two systems have the "same" trajectories, if you line them up according to $T$.

Now consider the case of an $n \times n$ real matrix $A$ that has a non-real eigenvalue $\lambda$. Since the non-real roots of the characteristic polynomial occur in conjugate pairs, $\bar{\lambda}$, the conjugate of $\lambda$ must also be an eigenvalue. Consider the generalized eigenspaces $G(\lambda)$ and $G(\bar{\lambda})$. As we have seen, $G(\lambda)$ is the nullspace of $(A-\lambda I)^n$, so if $v \in G(\lambda)$, $(A-\lambda I)^n v = 0$. Taking the conjugate of this equation yield $(\bar{A}-\bar{\lambda}I)^n \bar{v} = 0$. Since $A$ is real, $\bar{A} = A$, so we have $(A - \bar{\lambda}I)\bar{v} = 0$. Thus, $\bar{v} \in G(\bar{\lambda})$. Thus, conjugation maps $G(\lambda)$ into $G(\bar{\lambda})$ and, by the same argument $G(\bar{\lambda})$ into $G(\lambda)$. It is easy to see that these mappings are onto.

Since $\mathbb{C}^n$ is the direct sum of the generalized eigenspaces of $A$, we have

$$G(\lambda) \cap G(\bar{\lambda}) = \{0\}.$$

The only vector in $G(\lambda)$ with real entries is 0. To see this suppose $v \in G(\lambda)$ has real entries. Then $\bar{v} \in G(\bar{\lambda})$. But $\bar{v} = v$, so $v \in G(\lambda) \cap G(\bar{\lambda})$ and so $v = 0$.

Define $H = G(\lambda) \oplus G(\bar{\lambda})$. This space is closed under conjugation. Let $H_{\mathbb{R}} = H \cap \mathbb{R}^n$ be the set of real vectors in $H$.

Define a transformation $P \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$P(v) = \frac{1}{2}v + \frac{1}{2}\bar{v},$$

so $P(v)$ is the real part of $v$. Of course $P$ is not linear over the complex numbers, but it is linear over the real numbers. In other words, $P(v_1 + v_2) = P(v_1) + P(v_2)$ and $P(\alpha v) = \alpha P(v)$ for *real* scalars $\alpha$.

Since $H$ is closed under conjugation, $P(H) \subseteq H$, in fact $P(H) = H_{\mathbb{R}}$. We claim that $P$ maps $G(\lambda)$ one-to-one and onto $H_{\mathbb{R}}$. To show that $P$ is one-to-one on $G(\lambda)$, suppose that $v, w \in G(\lambda)$ and $P(v) = P(w)$. Since $P$ is linear, we have $P(v - w) = 0$, so it will suffice to show that $u = v - w$ is zero. Since $P(u) = 0$, we must have $\bar{u} = -u$. But $\bar{u} \in G(\bar{\lambda})$ and $-u \in G(\lambda)$, so we must have $0 = \bar{u} = -u$, so $u = 0$.

To prove that $P$ maps $G(\lambda)$ onto $H_{\mathbb{R}}$, let $w \in H_{\mathbb{R}}$. Since $w \in H$, we can write $w$ uniquely as $w = v_1 + v_2$ where $v_1 \in G(\lambda)$ and $v_2 \in G(\bar{\lambda})$. Since $w$ is real, we have $\bar{w} = w$. But then $v_1 + v_2 = w = \bar{v}_1 + \bar{v}_2$, so we must have $v_1 = \bar{v}_2$ and $v_2 = \bar{v}_1$. But now we have $2v_1 \in G(\lambda)$ and

$$\begin{aligned} P(2v_1) &= \frac{2v_1 + \overline{2v_1}}{2} \\ &= v_1 + \bar{v}_1 \\ &= v_1 + v_2 \\ &= w. \end{aligned}$$

We know that $G(\lambda)$ and $G(\bar{\lambda})$ are invariant under $A$, so $H$ is invariant under $A$. Since $A$ is a real matrix, $H_{\mathbb{R}}$ is invariant under $A$. Since $A$ is real, we have

$$\begin{aligned} P(Av) &= \frac{Av + \overline{Av}}{2} \\ &= \frac{Av + \bar{A}\bar{v}}{2} \\ &= \frac{Av + A\bar{v}}{2} \\ &= A\left(\frac{v+\bar{v}}{2}\right) \\ &= A(P(v)), \end{aligned}$$

so $A$ commutes with $P$. Thus, we have a commutative diagram

$$\begin{array}{ccc} G(\lambda) & \xrightarrow{A} & G(\lambda) \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ H_{\mathbb{R}} & \xrightarrow[A]{} & H_{\mathbb{R}} \end{array}$$

so, over the real numbers, $A$ behaves the same on $H_{\mathbb{R}}$ and $G(\lambda)$. (We could equally well have used $G(\bar{\lambda})$.)

If $A$ is real, let $\mu_1, \dots, \mu_r$ be the real eigenvalues and let $\lambda_1, \bar{\lambda}_1, \dots, \lambda_s, \bar{\lambda}_s$ be the conjugate pairs of non-real eigenvalues. Set $H(j) = G(\lambda_j) \oplus G(\bar{\lambda}_j)$. We then have

$$\mathbb{C}^n = G(\mu_1) \oplus \cdots \oplus G(\mu_r) \oplus H(1) \oplus \cdots \oplus H(s),$$

so it's not hard to see that

$$\mathbb{R}^n = G(\mu_1)_{\mathbb{R}} \oplus \cdots \oplus G(\mu_r)_{\mathbb{R}} \oplus H_{\mathbb{R}}(1) \oplus \cdots \oplus H_{\mathbb{R}}(s),$$

where, of course, $G(\mu_j)_{\mathbb{R}} = G(\mu_j) \cap \mathbb{R}^n$ and $H_{\mathbb{R}}(k) = H(k) \cap \mathbb{R}^n$. We know the behavior of the trajectories in $G(\mu_j)$ and so in $G(\mu_j)_{\mathbb{R}}$, and the behavior of the trajectories in $H_{\mathbb{R}}(k)$ is the same as the behavior of the trajectories in $G(\lambda_k)$ (or $G(\bar{\lambda}_k)$).

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