NOTES ON LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. Introduction

These notes provide an introduction to the theory of linear systems of differential equations, building on our previous sets of notes.

To introduce the subject, consider the following problem. We have two tanks of water. Tank 1 contains 100 gallons of water and Tank 2 contains 500 gallons of water. See Figure 1. Water is pumped from each tank into the other at a rate of 10 gallons per minute. At time $t = 0$, Tank 2 contains 200 lbs of dissolved salt. The problem is to find the time evolution of the amount of salt in both tanks. We assume that the tanks are well stirred, so we don't have to worry about the time it takes salt water from the inlet to reach the outlet in the tanks.

To find the equations, let y_1 be the number of pounds of salt in Tank 1, and let y_2 be the number of pounds of salt in Tank 2. The concentration of salt in Tank 2 is $y_2/500$ pound per gallon. Thus, the rate at which salt is entering Tank 1 from Tank 2 is $10(y_2/500)$ pound per minute. Similarly, the rate at which salt is leaving Tank 1 is $10(y_1/100)$ pounds per minute. Thus, we have

$$\frac{dy_1}{dt} = -\frac{y_1}{10} + \frac{y_2}{50}. \tag{1.1}$$

By similar reasoning,

$$\frac{dy_2}{dt} = \frac{y_1}{10} - \frac{y_2}{50}. \tag{1.2}$$

Adding these equations shows $d(y_1 + y_2)/dt = 0$, which reflects the fact that the amount of salt in the system is constant.

Equations (1.1) and (1.2) comprise a system of coupled linear differential equations. If we introduce the matrix notation

$$A = \begin{bmatrix} -1/10 & 1/50 \\ 1/10 & -1/50 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

we can write the system in matrix form as

$$\frac{dy}{dt} = Ay. \tag{1.3}$$

We want to find the vector valued function y that satisfies this matrix equation, and satisfies the initial condition

$$y(0) = \begin{bmatrix} 0 \\ 200 \end{bmatrix}.$$

Call the vector on the right y_0.

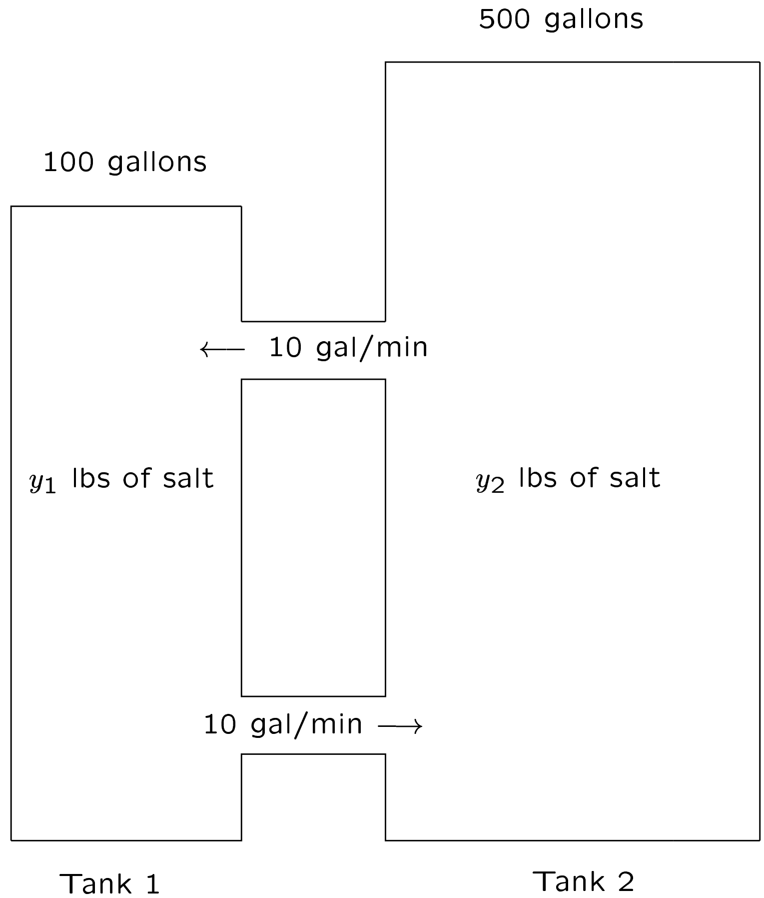

Figure 1. The Two Tanks

As a preview of the methods we will discuss, let's solve this problem. A calculation we already know how to do shows that the eigenvalues of the matrix A are 0 and $-3/25$, and if we let

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$$

then $P^{-1}AP = D$, where D is the diagonal matrix

$$D = \begin{bmatrix} -3/25 & 0 \\ 0 & 0 \end{bmatrix}.$$

To solve the system (1.3), introduce a new dependent variable z by $z = P^{-1}y$ Substituting $y = Pz$ in (1.3) gives us

$$\frac{d}{dt}(Pz) = P\frac{dz}{dt} = APz$$

and so

$$\frac{dz}{dt} = P^{-1}APz = Dz.$$

Written out this system is

$$\begin{aligned}\frac{dz_1}{dt} &= -\frac{3}{25}z_1\\ \frac{dz_2}{dt} &= 0.\end{aligned}$$

These equations are decoupled, so they are easy to solve. We must have

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-3t/25} \\ c_2 \end{bmatrix},$$

for constants c_1 and c_2. We want to choose the constants to satisfy the initial condition

$$z(0) = P^{-1}y_0 = \begin{bmatrix} -100/3 \\ 100/3 \end{bmatrix},$$

and so we have

$$z = \begin{bmatrix} -(100/3)e^{-3t/25} \\ 100/3 \end{bmatrix}.$$

Finally, the solution to the original problem is $y = Pz$, so

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{100}{3}\begin{bmatrix} 1 - e^{-3t/25} \\ 5 + e^{-3t/25} \end{bmatrix}.$$

Thus, as $t \to \infty$, the amount of salt in Tank 1 approaches 100/3 pounds and the amount in Tank 2 approaches 500/3 pounds. The graphs of y_1 and y_2 are given in Figure 2.

Of course, this solution depends on the fact that the matrix A is diagonalizable, since this is what allowed us to decouple the equations. We will later develop a method for finding the solution when the matrix is not diagonalizable.

2. General Theory of Linear Systems

In our example above, the coefficients of the linear system were constant. In this section, we will consider the more general case where the coefficients are allowed to be functions of t. Thus, we want to study the linear system

$$y'(t) = A(t)y(t) \tag{2.1}$$

where A is a continuous matrix valued function $J \to \mathbb{K}^{n\times n}$ and $J = (\alpha, \beta) \subseteq \mathbb{R}$ is some open interval.[1] We are seeking a vector-valued function $y(t)$ that satisfies (2.1).

The usual existence and uniqueness theorem for differential equations applies to (2.1). In the case of a nonlinear system, the solution might not exist over the entire time interval J, because the solution can run off to infinity before the end of the time interval. A simple example of this is the scalar initial value problem

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0.$$

Here $J = \mathbb{R}$. The solution is $y(t) = \tan(t)$ and $y(t) \to \infty$ as $t \to \pi/2$. The following theorem asserts in particular that this does not happen in the linear case.

[1] We use the symbol $\mathbb{K}$ to mean either $\mathbb{R}$ or $\mathbb{C}$, so a statement involving $\mathbb{K}$ is supposed to hold in either the real or complex case.

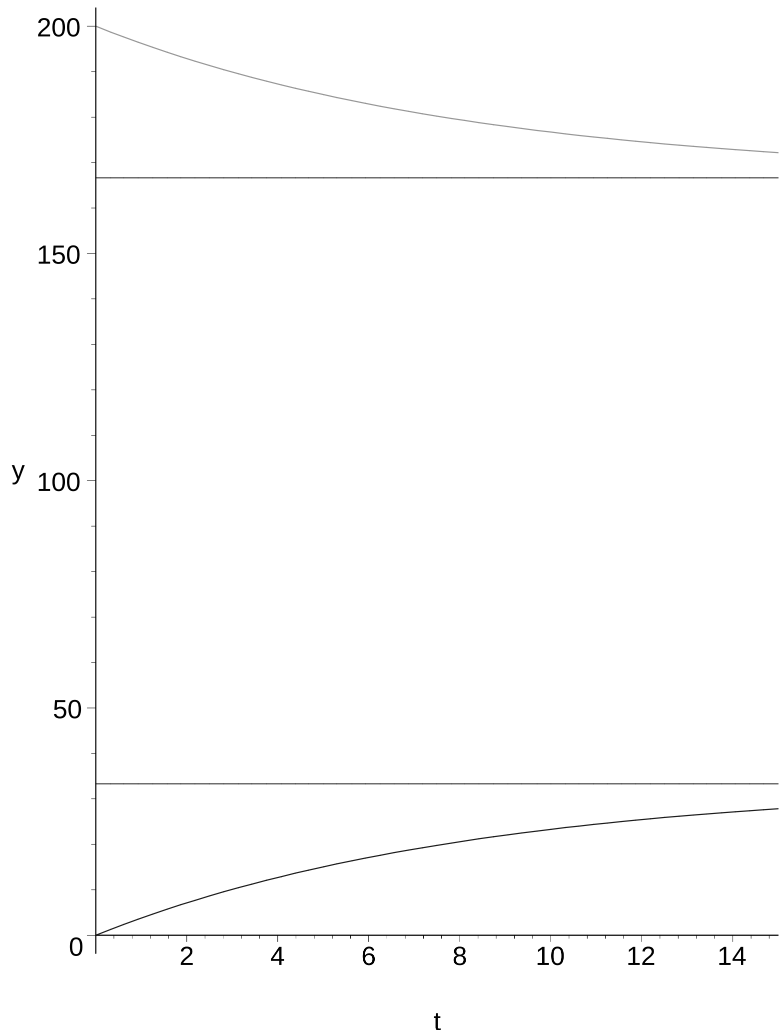

Figure 2. The graphs of y_1 (bottom) and y_2 (top)

Theorem 2.1. [Existence and Uniqueness Theorem] *Let $J \subseteq \mathbb{R}$ be an open interval and let $t \mapsto A(t)$ be a continuous matrix-valued function with values in $\mathbb{K}^{n\times n}$. Let $y_0 \in \mathbb{K}^n$ and $t_0 \in J$. Then there is a unique function $y\colon J \to \mathbb{K}^n\colon t \mapsto y(t)$ such that*

$$\begin{cases} y'(t) = A(t)y(t), & \text{for all } t \in J, \\ y(t_0) = y_0. \end{cases}$$

The proof of this theorem is, as they say, beyond the scope of the course.

Definition 2.2. Let $y_1, \dots, y_k$ be functions $J \to \mathbb{K}^n$, we say that $y_1, \dots, y_k$ are **linearly dependent on** J if there are constants $c_1, \dots, c_k$, not all zero, so that

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ky_k(t) = 0, \qquad \text{for all } t \in J.$$

If the functions are not linearly dependent, we say that are **linearly independent**.

Definition 2.3. If $y_1, \dots y_n$ are functions $J \to \mathbb{K}^n$, then the **Wronskian** of these functions is defined to be the function

$$W(t) = \det(y_1(t), \dots, y_n(t)),$$

where this notation means the determinant of the matrix whose columns are $y_1(t), \dots, y_n(t)$.

Proposition 2.4. *Let $y_1, \dots, y_n$ be functions $J \to \mathbb{K}^n$. If these functions are linearly dependent on J, then $W(t) = 0$ for all $t \in J$.*

Thus, if $W(t_0) \neq 0$ for some $t_0 \in J$, then $y_1, \dots, y_n$ are linearly independent.

Proof. If the functions are dependent, there are constants $c_1, \dots, c_n$, not all zero, so that

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0, \qquad \text{for all } t \in J.$$

Let $Y(t)$ be the matrix whose columns are $y_1(t), \dots, y_n(t)$ and let c be the column vector $c = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T$. Then, for each t, we have $Y(t)c = 0$. Since $c \neq 0$, this implies that the matrix $Y(t)$ is not invertible. Thus, we have $W(t) = \det(Y(t)) = 0$ for all $t \in J$. $\square$

Our next goal, which will take a little work is the following theorem. Recall that if A is a square matrix, the **trace** of A, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the diagonal elements, i.e., if A is $n \times n$,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

Theorem 2.5. [Abel's Theorem] *If $y_1, \dots, y_n$ are solutions of the linear system*

$$y'(t) = A(t)y(t), \qquad t \in J,$$

$W(t) = \det(y_1(t), \dots, y_n(t))$ is their Wronskian, and t_0 is a fixed point in J, then

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\, ds \right).$$

It follows that $W(t)$ is either zero for all $t \in J$, or $W(t)$ is never zero on J (in which case the solutions are linearly independent on J).

Exercise 2.6. Let A and B be $n \times n$ matrices. Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Hint: write out both sides as sums.

Exercise 2.7. If A and P are $n \times n$ matrices and P is invertible, show that $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(A)$. Thus, similar matrices have the same trace.

As the first step toward Abel's theorem, we need to deal with multilinear functions. Suppose that we have a function

$$F \colon \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \times \cdots \times \mathbb{K}^{n_k} \to \mathbb{K}^m.$$

We say that F is **multilinear** if it is linear in each slot separately, i.e. if you hold the vectors in all but one slot fixed, you get a linear function. In other words, if $v_i \in \mathbb{K}^{n_i}$, $w_j \in \mathbb{K}^{n_j}$ then

$$\begin{aligned}(2.2)\quad & F(v_1, \dots, v_{j-1}, \alpha v_j + \beta w_j, v_{j+1}, \dots v_n) = \\ & \qquad \alpha F(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots) + \beta F(v_1, \dots, v_{j-1}, w_j, v_{j+1}, \dots, v_n).\end{aligned}$$

For example, the determinant is multilinear as a function of the columns of the matrix.

Proposition 2.8. *Let $v_j \colon J \to \mathbb{K}^{n_j}$, $j = 1, \dots, k$, be differentiable vector-valued functions and let*

$$F \colon \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \times \cdots \times \mathbb{K}^{n_k} \to \mathbb{K}^m$$

be a multilinear function. Then

$$\frac{d}{dt} F(v_1(t), \dots, v_k(t)) = \sum_{j=1}^{k} F(v_1(t), \dots, v_{j-1}(t), v_j'(t), v_{j+1}(t), \dots v_k(t)).$$

Proof. For simplicity, consider the case where

$$F \colon \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \to \mathbb{K}^m$$

is multilinear. The general case is the same, just notationally messier.

If $a \in \mathbb{K}^{n_1}$, we can write $a = \sum_{i=1}^{n_1} a_i e_i$ where the e_i's are the standard basis vectors in $\mathbb{K}^{n_1}$. Similarly, if $b \in \mathbb{K}^{n_2}$, $b = \sum_{j=1}^{n_2} b_j e_j$. Then we have

$$\begin{aligned} F(a, b) &= F\left(\sum_{i=1}^{n_1} a_i e_i, b\right) \\ &= \sum_{i=1}^{n_1} a_i F(e_i, b) \\ &= \sum_{i=1}^{n_1} a_i F\left(e_i, \sum_{j=1}^{n_2} b_j e_j\right) \\ &= \sum_{i=1}^{n_1} a_i \sum_{j=1}^{n_2} b_j F(e_i, e_j) \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_i b_j F(e_i, e_j) \end{aligned}$$

where the $F(e_i, e_j)$'s are constant vectors.

Thus, if a and b are functions of t, we have

$$F(a(t), b(t)) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_i(t) b_j(t) F(e_i, e_j).$$

Since the $F(e_i, e_j)$'s are constant, the usual product rule from calculus gives us

$$\begin{aligned} \frac{d}{dt} F(a(t), b(t)) &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [a_i'(t) b_j(t) + a_i(t) b_j'(t)] F(e_i, e_j) \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_i'(t) b_j(t) F(e_i, e_j) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_i(t) b_j'(t) F(e_i, e_j) \\ &= F(a'(t), b(t)) + F(a(t), b'(t)). \end{aligned}$$

□

One important application to of this theorem is to take F to be the function $\mathbb{K}^{m \times p} \times \mathbb{K}^{p \times n} \to \mathbb{K}^{n \times n}$ given by matrix multiplication. Thus, if $A(t)$ and $B(t)$ are differentiable matrix-valued functions,

$$\frac{d}{dt} A(t) B(t) = A'(t) B(t) + A(t) B'(t).$$

The immediate application we want to make is to the determinant function. Thus, if $y_1, \dots, y_n$ are functions with values in $\mathbb{K}^n$, then

$$\frac{d}{dt}\det(y_1(t), \dots, y_n(t)) = \sum_{j=1}^{n} \det(y_1(t), \dots, y_{j-1}(t), y_j'(t), y_{j+1}(t), \dots, y_n(t)).$$

Next, we need another definition.

Definition 2.9. A multilinear function

$$F\colon \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n \text{ factors}} \to \mathbb{K}$$

is called **skew symmetric** if exchanging the vectors in two slots changes the sign. In other words,

$$F(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots v_{k-1}, v_k, v_{k+1}, \dots, v_n) = \\ -F(v_1, \dots, v_{j-1}, v_k, v_{j+1}, \dots v_{k-1}, v_j, v_{k+1}, \dots, v_n). \tag{2.3}$$

An obvious example is the determinant function, considered as a function of the columns of the matrix. In fact, there are not very many functions like this.

Proposition 2.10. *Let*

$$F\colon \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n \text{ factors}} \to \mathbb{K}$$

be a skew symmetric multilinear function. Then

$$F(v_1, \dots, v_n) = k \det(v_1, \dots, v_n)$$

for all vectors $v_1, \dots, v_n$. The constant k is given by

$$k = F(e_1, \dots, e_n),$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{K}^n$.

Proof. Given vectors $v_1, \dots, v_n$, we can write

$$v_j = \sum_{i=1}^{n} e_i v_{ij}.$$

Using the multilinearity of F, we can expand $F(v_1, \dots, v_n)$ as

$$F(v_1, v_2, \dots, v_n) = \sum_{j_1, \dots, j_n = 1}^{n} v_{j_1 1} v_{j_2 2} \dots v_{j_n n} F(e_{j_1}, e_{j_2}, \dots, e_{j_n}).$$

Similarly,

$$\det(v_1, v_2, \dots, v_n) = \sum_{j_1, \dots, j_n = 1}^{n} v_{j_1 1} v_{j_2 2} \dots v_{j_n n} \det(e_{j_1}, e_{j_2}, \dots, e_{j_n}).$$

Thus, it will suffice to show that

$$F(e_{j_1}, e_{j_2}, \dots, e_{j_n}) = k \det(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \tag{2.4}$$

for all choices of the indices $j_1, j_2, \dots, j_n$, where k is defined as in the proposition.

If there is a repeated index among $j_1, \dots, j_n$, then (2.4) holds, since in this case both sides have a repeated vector and hence are zero, by skew symmetry.

Thus, it remains to consider the case where $j_1, \dots, j_n$ is a permutation of $1, \dots, n$. We have

$$F(e_1, e_2, \dots, e_n) = k \det(e_1, e_2, \dots, e_n)$$

by the definition of k and the fact that the determinant of the identity matrix is 1. If we apply a transposition of two indices to both sides of the last equation, we will still have equality, since both sides will change by a minus sign. Similarly, if we apply a sequence of transpositions, we will still have equality. But any permutation can be obtained by a sequence of transpositions, so we can conclude that (2.4) holds for any permutation. This completes the proof. □

Proposition 2.11. *Let A be an $n \times n$ matrix with entries in $\mathbb{K}$ and define*

$$F\colon \underbrace{\mathbb{K}^n \times \cdots \times \mathbb{K}^n}_{n \text{ factors}} \to \mathbb{K}$$

by

$$F(v_1, \dots, v_n) = \sum_{j=1}^{n} \det(v_1, \dots, v_{j-1}, Av_j, v_{j+1}, \dots v_n).$$

Then

$$F(v_1, \dots, v_n) = \operatorname{tr}(A) \det(v_1, \dots, v_n).$$

Proof. It's not too hard to see that F is skew symmetric. For example, in the case $n = 3$, we have

$$F(v_1, v_2, v_3) = \det(Av_1, v_2, v_3) + \det(v_1, Av_2, v_3) + \det(v_1, v_2, Av_3).$$

If we switch slots 1 and 3, we have

$$\begin{aligned} F(v_3, v_2, v_1) &= \det(Av_3, v_2, v_1) + \det(v_3, Av_2, v_1) + \det(v_3, v_2, Av_1) \\ &= -\det(v_1, v_2, Av_3) - \det(v_1, Av_2, v_3) - \det(Av_1, v_2, v_3) \\ &= -F(v_1, v_2, v_3). \end{aligned}$$

Since F is skew symmetric, we must have

$$F(v_1, \dots, v_n) = k \det(v_1, \dots, v_n),$$

where

$$k = F(e_1, e_2, \dots, e_n).$$

But then,

$$k = \sum_{j=1}^{n} \det(e_1, \dots, e_{j-1}, Ae_j, e_{j+1}, \dots, e_n).$$

Consider the jth term in this sum. We have

$$Ae_j = \sum_{k=1}^{n} e_k a_{kj}$$

and so

$$\begin{aligned} \det(e_1, \dots, e_{j-1}, Ae_j, e_{j+1}, \dots, e_n) &= \det(e_1, \dots, e_{j-1}, \sum_{k=1}^{n} e_k a_{kj}, v_{j+1}, \dots, v_n) \\ &= \sum_{k=1}^{n} a_{kj} \det(e_1, \dots, e_{j-1}, e_k, e_{j+1}, \dots, e_n). \end{aligned}$$

But all of the terms in the last sum with $k \neq j$ are zero because of a repeated vector. The term with $k = j$ just comes out to a_{jj}. Thus, we have

$$\begin{aligned} k &= F(e_1, \dots, e_n) \\ &= \sum_{j=1}^{n} \det(e_1, \dots, e_{j-1}, Ae_j, e_{j+1}, \dots, e_n) \\ &= \sum_{j=1}^{n} a_{jj} \\ &= \operatorname{tr}(A). \end{aligned}$$

This completes the proof. □

We're now ready to prove Abel's Theorem (Theorem 2.5). So, let $y_1, \dots, y_n$ be functions with values in $\mathbb{K}^n$ which are solutions of

$$y'(t) = A(t)y(t), \qquad t \in J, \tag{2.5}$$

where $A(t)$ is a given continuous function with values in $\mathbb{K}^{n\times n}$. Let

$$W(t) = \det(y_1(t), y_2(t), \dots, y_n(t))$$

be the Wronskian of these solutions. By the machinery we have just developed, we have

$$\begin{aligned} \frac{d}{dt}W(t) &= \frac{d}{dt}\det(y_1(t), \dots, y_n(t)) \\ &= \sum_{j=1}^{n} \det(y_1(t), \dots, y_{j-1}(t), y_j'(t), y_{j+1}(t), \dots, y_n(t)) \\ &= \sum_{j=1}^{n} \det(y_1(t), \dots, y_{j-1}(t), A(t)y_j(t), y_{j+1}(t), \dots, y_n(t)) \\ &= \operatorname{tr}(A(t)) \det(y_1(t), \dots, y_n(t)) \\ &= \operatorname{tr}(A(t))W(t). \end{aligned}$$

Thus, $W(t)$ is a solution of the scalar linear differential equation

$$W'(t) = \operatorname{tr}(A(t))W(t). \tag{2.6}$$

To solve this equation, we use an integrating factor. Choose $t_0 \in J$ and define

$$\alpha(t) = \int_{t_0}^{t} \operatorname{tr}(A(s))\, ds$$

so $\alpha'(t) = \operatorname{tr}(A(t))$ (by the fundamental theorem of calculus) and $\alpha(t_0) = 0$. Now, rewrite (2.6) as

$$W'(t) - \operatorname{tr}(A(t))W(t) = 0.$$

Multiply both sides of this equation by $e^{-\alpha(t)}$ to get

$$e^{-\alpha(t)}W(t) - e^{-\alpha(t)} \operatorname{tr}(A(t))W(t) = 0, \tag{2.7}$$

The left-hand side of this equation is

$$\frac{d}{dt}\big(e^{-\alpha(t)}W(t)\big),$$

so (2.7) is the same as

$$\frac{d}{dt}\big(e^{-\alpha(t)}W(t)\big) = 0.$$

From this we conclude

$$e^{-\alpha(t)}W(t) = C,$$

where C is a constant. Setting $t = t_0$ shows that $C = W(t_0)$. Thus, we conclude that the solution of (2.6) is

$$W(t) = W(t_0)e^{\alpha(t)} = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))\,ds\right).$$

This completes the proof of Abel's Theorem.

Exercise 2.12. Consider the nth order scalar linear differential equation

$$a_n y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0, \tag{2.8}$$

where the a_j's are constants with $a_n \neq 0$. Show that if y is an n-times differentiable scalar function that is a solution of (2.8), then vector-valued function

$$Y(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

is a solution of the vector differential equation

$$Y'(t) = AY(t), \tag{2.9}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & -a_3/a_n & \dots & -a_{n-1}/a_n \end{bmatrix}.$$

Conversely, if

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

is a solution of (2.9) then y_1 is a solution of (2.8).

Let $y_1, \dots, y_n$ be solutions of (2.8) and let

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

be their Wronskian. Use Abel's theorem to show that

$$W(t) = W(0)e^{-a_{n-1}t/a_n}.$$

Now consider again the differential equation

$$y'(t) = A(t)y(t), \qquad t \in J \tag{2.10}$$

where $A(t)$ is a given continuous function with values in $\mathbb{K}^{n\times n}$.

A differentible matrix valued function $\Phi(t)$ on J is a **solution matrix** of (2.10) if

$$\Phi'(t) = A(t)\Phi(t), \qquad t \in J.$$

This is the same as saying that each column of $\Phi(t)$ is a solution of (2.10).

Proposition 2.13. *If $A(t)$ is a given continuous function on J with values in $\mathbb{K}^{n\times n}$, $t_0 \in J$ and $C \in \mathbb{K}^{n\times n}$ is a constant matrix, there is a unique $\mathbb{K}^{n\times n}$-valued function $\Phi(t)$ that solves the initial value problem*

$$\begin{cases} \Phi'(t) = A(t)\Phi(t), & t \in J, \\ \Phi(t_0) = C. \end{cases} \tag{2.11}$$

Proof. If we use $\varphi_1(t), \dots, \varphi_n(t)$ for the columns of $\Phi(t)$ and $c_1, \dots, c_n$ for the columns of C, then (2.11) is the same as

$$\varphi_j'(t) = A(t)\varphi_j(t), \quad \varphi_j(t_0) = c_j$$

for $j = 1, \dots, n$. These initial value problems have a unique solution by the existence and uniqueness theorem for (2.10), and so (2.11) has a unique solution. □

Definition 2.14. A solution matrix $\Phi(t)$ for (2.10) is a **fundamental matrix** for (2.10) if $\det(\Phi(t)) \neq 0$ for all $t \in J$. This is the same as saying that the columns of $\Phi(t)$ are n linearly independent solutions of (2.10).

Proposition 2.15. *There is a fundamental matrix for* (2.10).

Proof. Take $\Phi(t)$ to be the solution of the initial value problem (2.11) where C is some nonsingular matrix. We then have $\det(\Phi(t_0)) = \det(C) \neq 0$, so $\det(\Phi(t)) \neq 0$ for all t by Abel's Theorem. □

Proposition 2.16. *Let $\Phi(t)$ be a fundamental matrix for* (2.10)*, and let $\Psi(t)$ be another matrix-valued function on J. Then $\Psi(t)$ is a fundamental matrix if and only if*

$$\Psi(t) = \Phi(t)C, \qquad t \in J.$$

where C is an invertible $n \times n$ matrix.

Proof. If C is invertible, $\Phi(t)C$ is invertible for all t and

$$\frac{d}{dt}[\Phi(t)C] = \Phi'(t)C = A\Phi(t)C = A[\Phi(t)C],$$

so $\Phi(t)C$ is a fundamental matrix.

Suppose that $\Psi(t)$ is another fundamental matrix. Choose $t_0 \in J$. Then we have $\Psi(t_0) = \Phi(t_0)C$, where $C = \Phi(t_0)^{-1}\Psi(t_0)$ is invertible. Thus, $\Psi(t)$ and $\Phi(t)C$ are solution matrices for (2.10) that satisfy the same initial condition at t_0. Hence $\Psi(t) = \Phi(t)C$ for all t. □

Proposition 2.17. *Let $\Phi(t)$ be a fundamental matrix for*

$$y'(t) = A(t)y(t), \qquad t \in J.$$

Then the solution of the initial value problem

$$\begin{cases} y'(t) = A(t)y(t), & t \in J \\ y(t_0) = y_0 \end{cases} \tag{2.12}$$

is

$$y(t) = \Phi(t)\Phi(t_0)^{-1}y_0.$$

The proof of this proposition is left as a simple exercise.

Thus, once we know a fundamental matrix, we can solve the initial value problem (2.12) for any initial condition.

As a final topic in this section, we will discuss how to solve the linear *inhomogeneous* initial value problem

$$\begin{cases} y'(t) = A(t)y(t) + b(t) \\ y(t_0) = y_0, \end{cases} \tag{2.13}$$

where $b(t)$ is some given vector-valued function. In the applications $b(t)$ is usually a force or influence from the outside.

One way to do this is variation of parameters. Let $\Phi(t)$ be a fundamental matrix for the homogeneous linear equation

$$y'(t) = A(t)y(t)$$

and seek a solution of (2.13) of the form $y(t) = \Phi(t)v(t)$ where $v(t)$ is a vector-valued function to be determined. We have

$$\begin{aligned} y'(t) &= \Phi'(t)v(t) + \Phi(t)v'(t) \\ &= A(t)\Phi(t)v(t) + \Phi(t)v'(t). \end{aligned}$$

Plugging into both sides of the equation in (2.13), we have

$$A(t)\Phi(t)v(t) + \Phi(t)v'(t) = A(t)\Phi'(t)v(t) + b(t)$$

which reduces to

$$\Phi(t)v'(t) = b(t),$$

or

$$v'(t) = \Phi(t)^{-1}b(t).$$

Hence, we must have

$$v(t) = c + \int_{t_0}^{t} \Phi(s)^{-1}b(s)\,ds,$$

where c is some constant vector. Thus, so far,

$$y(t) = \Phi(t)v(t) = \Phi(t)c + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}b(s)\,ds.$$

Setting $t = t_0$ shows that $y(t_0) = \Phi(t_0)c$, so to satisfy the initial condition $y(t_0) = y_0$, we must choose $c = \Phi(t_0)^{-1}y_0$. Thus, we get the following formula for the solution of the initial value problem (2.13):

$$y(t) = \Phi(t)\Phi(t_0)^{-1}y_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}b(s)\,ds, \tag{2.14}$$

which is known as **the variation of parameters formula**.

3. Systems of Linear Differential Equations with Constant Coefficients

In this section, we consider the case of a linear system of differential equations where the coefficient matrix is constant. Thus, the equation looks like

$$y'(t) = Ay(t) \tag{3.1}$$

for a constant matrix A. In this case it is possible to be much more explicit than in the case of nonconstant coefficients.

Since A is defined for all time, all of the solutions of (3.1) can be extended to the whole real axis.

As we saw in the last section, what we need to find is a fundamental matrix for (3.1). We know that given any matrix B and time t_0, there is a unique matrix valued function Ψ such that

$$\Psi'(t) = A\Psi(t), \quad \Psi(t_0) = B. \tag{3.2}$$

If B is invertible, Ψ will be a fundamental matrix. It is convenient to fix the time and the initial condition. Thus, if $A \in \mathbb{K}^{n\times n}$, let Φ_A be the matrix-valued function that satisfies

$$\Phi_A'(t) = A\Phi_A(t), \quad \Phi_A(0) = I. \tag{3.3}$$

We can develop some useful properties of Φ_A from the defining differential equations.

Proposition 3.1. *Suppose that $A \in \mathbb{K}^{n\times n}$. Then the following statements are true.*

(1) *If $B \in \mathbb{K}^{n\times n}$ and $BA = AB$ then $B\Phi_A(t) = \Phi_A(t)B$ for all $t \in \mathbb{R}$.*
(2) *If $BA = BA$ then*
$$\Phi_A(t)\Phi_B(t) = \Phi_{A+B}(t) = \Phi_B(t)\Phi_A(t), \qquad t \in \mathbb{R}.$$
(3) *$\Phi_A(t+s) = \Phi_A(t)\Phi_A(s)$ for all $t, s \in \mathbb{R}$.*
(4) *$\Phi_A(t)^{-1} = \Phi_A(-t)$, for all $t \in \mathbb{R}$.*
(5) *If P is an invertible $n \times n$ matrix then*
$$P^{-1}\Phi_A(t)P = \Phi_{P^{-1}AP}(t), \qquad t \in \mathbb{R}.$$

Proof. (1) Let $\Psi_1(t) = \Phi_A(t)B$. Then

$$\begin{aligned}\Psi_1'(t) &= \Phi_A'(t)B\\ &= A\Phi_A B\\ &= A\Psi_1(t).\end{aligned}$$

Now let $\Psi_2(t) = B\Phi_A(t)$. Then

$$\begin{aligned}\Psi_2'(t) &= B\Phi_A'(t)\\ &= BA\Phi_A(t)\\ &= AB\Phi_A(t) && \text{because } AB = BA\\ &= A\Psi_2(t).\end{aligned}$$

Thus, Ψ_1 and Ψ_2 satisfy the same differential equation. We have

$$\Psi_1(0) = \Phi_A(0)B = IB = B = B\Phi_A(0) = \Psi_2(0),$$

so Ψ_1 and Ψ_2 satisfy the same initial condition. Thus, we must have $\Psi_1 = \Psi_2$.

(2) Let $\Psi(t) = \Phi_A(t)\Phi_B(t)$. We then have $\Psi(0) = I$ and

$$\begin{aligned} \Psi'(t) &= \Phi_A'(t)\Phi_B(t) + \Phi_A(t)\Phi_B'(t) \\ &= A\Phi_A(t)\Phi_B(t) + \Phi_A(t)B\Phi_B(t) \\ &= A\Phi_A(t)\Phi_B(t) + B\Phi_A(t)\Phi_B(t) && \text{by part (1),} \\ &= (A+B)\Phi_A(t)\Phi_B(t) \\ &= (A+B)\Psi(t). \end{aligned}$$

Thus, Ψ satisfies the same differential equation and initial condition as Φ_{A+B}, so we have $\Phi_{A+B}(t) = \Phi_A(t)\Phi_B(t)$. The same argument with the roles of A and B reversed completes the proof.

(3) Fix s and let $\Psi_1(t) = \Phi_A(t)\Phi_A(s)$. It is then easy to check

$$\Psi_1'(t) = A\Psi_1(t), \quad \Psi_1(0) = \Phi_A(s).$$

On the other hand, define $\Psi_2(t) = \Phi_A(t+s)$. Then,

$$\Psi_2'(t) = \Phi_A'(t+s) = A\Phi_A(t+s) = A\Psi_2,$$

(using the chain rule) and $\Psi_2(0) = \Phi_A(0+s) = \Phi_A(s)$. Thus, Ψ_1 and Ψ_2 satisfy the same differential equation and initial condition, so $\Psi_1 = \Psi_2$.

(4) This follows from the last part:

$$\Phi_A(-t)\Phi_A(t) = \Phi_A(-t+t) = \Phi_A(0) = I = \Phi_A(t)\Phi_A(-t),$$

so $\Phi_A(t)^{-1} = \Phi_A(t)$

(5) Let $B = P^{-1}AP$ and set $\Psi(t) = P^{-1}\Phi_A(t)P$. Then $\Psi(0) = I$ and

$$\begin{aligned} \Psi'(t) &= \frac{d}{dt}P^{-1}\Phi_A(t)P \\ &= P^{-1}\Phi_A'(t)P \\ &= P^{-1}A\Phi_A(t)P \\ &= P^{-1}APP^{-1}\Phi_A(t)P \\ &= B\Psi(t) \end{aligned}$$

Thus, we must have $\Psi = \Phi_B$.

$\square$

We've used the notation $\Phi_A(t)$ up until now because we wanted to emphasize that its properties can be deduced from the defining initial value problem. Most people use the notation

$$\Phi_A(t) = e^{tA}.$$

The right-hand side is the exponential function of a matrix. One way to define the exponential of a matrix is to use the differential equation, as we have done. Another approach is to use an infinite series. Recall from Calculus that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots,$$

where the series has infinite radius of convergence. By analogy, we can substitute $x = tA$ in this series to get

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \frac{t^4}{4!} A^4 + \cdots, \tag{3.4}$$

where each term in the series is an $n \times n$ matrix. It can be shown that this series converges for any matrix A, meaning that each entry in the $n \times n$ matrix on the right converges to the corresponding entry of the sum. Substituting $t = 0$ in (3.4) shows that $e^{0A} = I$. It can be shown that it is valid to differentiate the series term by term, which gives

$$\begin{aligned} \frac{d}{dt} e^{tA} &= \frac{d}{dt}\big[I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \frac{t^4}{4!} A^4 + \cdots\big] \\ &= 0 + A + tA^2 + \frac{t^2}{2} A^3 + \frac{t^3}{3!} A^4 + \cdots \\ &= A\big[I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \cdots\big] \\ &= Ae^{tA}. \end{aligned}$$

Thus, e^{tA} as defined by the series, is a solution of the initial value problem that defines $\Phi_A(t)$, so $\Phi_A(t) = e^{tA}$.

We won't justify this computation here, because there really isn't anything about e^{tA} that you can get from the series that you can't get from the differential equation. Computing e^{tA} directly from the series isn't easy in the general case, since you don't have a closed form for the entries of A^k, $k = 1, 2, \ldots$.

The properties in Proposition 3.1 look nicer in exponential notation and, in some case, look like laws for the scalar exponential function. Here are the statements.

Proposition 3.2. *If A is an $n \times n$ matrix with entries in $\mathbb{K}$, the following statements are true.*

1. *If $AB = BA$ then $Be^{tA} = e^{tA}B$ for all t.*
2. *If $AB = BA$ then*
$$e^{tA}e^{tB} = e^{t(A+B)} = e^{tB}e^{tA}.$$
3. *For all $t, s \in \mathbb{R}$,*
$$e^{tA}e^{sA} = e^{(t+s)A} = e^{sA}e^{tA}.$$
4. *For all $t \in \mathbb{R}$,*
$$(e^{tA})^{-1} = e^{-tA}.$$
5. *If P is an invertible $n \times n$ matrix,*
$$P^{-1}e^{tA}P = e^{tP^{-1}AP}.$$

We now want to consider how to compute e^{tA}. The easiest case, as we saw in the introduction, is if A is diagonalizable. Suppose that

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

is a diagonal matrix. The λ_j's are, of course, the eigenvalues of D, repeated according to their multiplicities.

If we write out the system of differential equations

$$\frac{dy}{dt} = Dy$$

we get

$$\begin{aligned}\frac{dy_1}{dt} &= \lambda_1 y_1\\ \frac{dy_2}{dt} &= \lambda_2 y_2\\ &\vdots\\ \frac{dy_n}{dt} &= \lambda_n y_n.\end{aligned}$$

Since the equations are decoupled, they are easy to solve individually, so we have the general solution

$$y = \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix} c_1e^{\lambda_1 t}\\ c_2e^{\lambda_2 t}\\ \vdots\\ c_ne^{\lambda_n t}\end{bmatrix} \tag{3.5}$$

and setting $t = 0$ gives

$$y(0) = \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}. \tag{3.6}$$

To find $\Phi(t) = \Phi_D(t) = e^{Dt}$, we have to solve the matrix initial value problem

$$\frac{d}{dt}\Phi(t) = D\Phi(t), \quad \Phi(0) = I. \tag{3.7}$$

If we denote the jth column of $\Phi(t)$ by $\Phi_j(t)$, the matrix initial value problem above is equivalent to the n vector initial value problems

$$\frac{d}{dt}\Phi_j(t) = D\Phi_j(t), \quad \Phi_j(0) = e_j, \tag{3.8}$$

$j = 1, 2, \dots, n$, where $e_1, \dots, e_n$ are the standard basis vectors of $\mathbb{K}^n$. From (3.5) and (3.6), we see that the solution of the initial value problem (3.8) is

$$\Phi_j(t) = e^{\lambda_j t}e_j$$

and hence that the solution to (3.7) is

$$e^{Dt} = \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \dots & 0\\ 0 & e^{\lambda_2 t} & 0 & \dots & 0\\ 0 & 0 & e^{\lambda_3 t} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & e^{\lambda_n t}\end{bmatrix},$$

which is again a diagonal matrix.

Now consider finding e^{At} where A is diagonalizable. We can find an invertible matrix P so that $P^{-1}AP = D$, where D is diagonal. The diagonal entries in D are, of course, the eigenvalues of A. We know what the matrix e^{Dt} is from the computation above. We have $A = PDP^{-1}$, and so we have

$$e^{At} = e^{tPDP^{-1}} = Pe^{tD}P^{-1},$$

from Proposition 3.2. Thus, we know how to compute e^{tA}, and we can say that the entries in e^{At} are linear combinations with constant coefficients of the functions $e^{\lambda_j t}$, where the λ_j's are the eigenvalues of A.

What happens when A is not diagonalizable? Recall that A can be written uniquely as $A = S + N$, where S is diagonalizable, N is nilpotent and $SN = NS$. The eigenvalues of A are the same as the eigenvalues of S. Since S and N commute, we have

$$e^{At} = e^{St+Nt} = e^{St}e^{Nt}, \tag{3.9}$$

from Proposition 3.2. Since S is diagonalizable, we know how to compute e^{St}. To handle e^{Nt}, we have the next proposition.

Proposition 3.3. *Let N be a nilpotent matrix. Suppose that $N^k \neq 0$, but $N^{k+1} = 0$. Then*

$$e^{Nt} = I + tN + \frac{1}{2!}t^2N^2 + \frac{1}{3!}t^3N^3 + \cdots + \frac{1}{k!}t^kN^k. \tag{3.10}$$

Hence, each entry of e^{tN} is a polynomial in t.

Proof. The formula (3.10) is what you get if you plug N into the series (3.4), since $N^p = 0$ for $p > k$. If the series makes you nervous, we can also verify the formula (3.10) using the defining initial value problem.

Define $\Psi(t)$ by

$$\Psi(t) = I + tN + \frac{1}{2!}t^2N^2 + \frac{1}{3!}t^3N^3 + \cdots + \frac{1}{k!}t^kN^k.$$

Differentiating, we get

$$\Psi'(t) = N + tN^2 + \frac{1}{2!}t^2N^3 + \ldots \frac{1}{(k-1)!}t^{k-1}N^k.$$

On the other hand, we have

$$\begin{aligned} N\Psi(t) &= N(I + tN + \frac{1}{2!}t^2N^2 + \frac{1}{3!}t^3N^3 + \cdots + \frac{1}{k!}t^kN^k) \\ &= N + tN^2 + \frac{1}{2!}t^2N^3 + \frac{1}{3!}N^4 + \cdots + \frac{1}{(k-1)!}t^{k-1}N^k + \frac{1}{k!}t^kN^{k+1} \\ &= N + tN^2 + \frac{1}{2!}t^2N^3 + \frac{1}{3!}N^4 + \cdots + \frac{1}{(k-1)!}t^{k-1}N^k, \end{aligned}$$

since $N^{k+1} = 0$. Thus, we have $\Psi'(t) = N\Psi(t)$. We also clearly have $\Psi(0) = I$, so we can conclude $\Psi(t) = e^{Nt}$

□

Exercise 3.4. Find e^{tN} where

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Generalize to the case where N is $n \times n$ with 1's on the superdiagonal and all other entries equal to zero.

If we go back to equation (3.9), we now know how to compute both e^{St} and e^{Nt}, and hence $e^{At} = e^{St}e^{Nt}$. In particular, we can conclude that every entry of e^{At} is a sum of functions of the form $p(t)e^{\lambda t}$, where $p(t)$ is a polynomial in t and λ is an eigenvalue of A.

Exercise 3.5. If J is in Jordan Canonical Form, show how to compute e^{Jt} explicitly.

Although the above theoretically gives a method of computing e^{At}, it's not very easy to use for explicit computations, since it is not easy to find the decomposition $A = S + N$.

Computing e^{tA} numerically (i.e., taking care with round-off error and dealing with large matrices) is a non-trivial problem. See a numerical guru.[2]

We will give a method which works when you can find the eigenvalues and their multiplicities and the matrix is not too large. We are following a paper by I.E. Leonard[3] so we'll call this Leonard's algorithm. The formula is not originally due to Leonard, but this approach is his.

We need to recall quickly how to solve higher order, scalar, linear, homogeneous differential equations with constant coefficients. In other words, we want to find the solutions of equations of the form

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0, \tag{3.11}$$

where $a_0, \ldots, a_n$ are constants, with $a_n \neq 0$ and $y(t)$ is a function with values in $\mathbb{K}$. Usually one divides through by a_n, but we'll leave it in our equation. The polynomial

$$p(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$$

is called the **characteristic polynomial** of (3.11). Equation (3.11) can be written as

$$p(D)y = 0$$

where D stands for the differential operator d/dt.

To find the general solution of (3.11), we need a fundamental set of solutions, i.e., a collection $y_1, \ldots, y_n$ of functions that are solutions of (3.11) and are linearly independent. To check if the solutions are linearly independent, we can use their Wronskian.

Suppose that $y_1, \ldots, y_n$ are linearly dependent functions. Then, there are constants $c_1, \ldots, c_n$, not all zero, such that

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0, \qquad \text{for all } t.$$

---

[2] A good place to start is the well known paper "Nineteen dubious ways to compute the exponential of a matrix", by Cleve Moler and Charles Van Loan (*SIAM Rev*, **20**(1978), no. 4, 801–836)

[3] "The matrix exponential," *SIAM Rev*, **38**(1996), no. 3, 507–512

If we differentiate this equation repeatedly, we have

$$\begin{gathered} c_1 y_1'(t) + c_2 y_2'(t) + \cdots + c_n y_n'(t) = 0 \\ c_1 y_1''(t) + c_2 y_2''(t) + \cdots + c_n y_n''(t) = 0 \\ \vdots \\ c_1 y_1^{(n-1)}(t) + c_2 y_2^{(n-1)}(t) + \cdots + c_n y_n^{(n-1)}(t) = 0, \end{gathered}$$

for all t. If we let c be the column vector $c = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T$ and let $M(t)$ be the Wronskian matrix of the functions, i.e.,

$$M(t) = \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ y_1''(t) & y_2''(t) & \dots & y_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{bmatrix}$$

we see that $M(t)c = 0$ for all t. Since $c \neq 0$, we conclude that $\det(M(t)) = 0$ for all t. The function $W(t) = \det(M(t))$ is called the Wronskian of the functions $y_1, \dots, y_n$.

Thus, we conclude that the Wronskian $W(t)$ is identically zero if the functions are dependent. Hence, if W is nonzero at one point, the functions are independent.

If $y_1, \dots, y_n$ are solutions of (3.11), Abel's Theorem shows that the Wronskian is either zero for all t, or non-zero for all t. (See Exercise 2.12.)

We can find a fundamental set of solutions of (3.11) if we can find the roots of the characteristic polynomial $p(\lambda)$ and their multiplicities. We recall the facts in the next proposition.

Proposition 3.6. *If r is a root of the characteristic polynomial $p(\lambda)$ with multiplicity m, then r contributes the functions*

$$e^{rt}, te^{rt}, \dots, t^{m-1}e^{rt}$$

to the fundamental set of solution of (3.11). *Since the multiplicities of the roots add up to n, we get n linearly independent solutions in this way. Thus, a root r of multiplicity* 1 *contributes the single function e^{rt} to the fundamental set.*

In the case where the coefficients of $p(\lambda)$ are all real, it is often desirable to write down the real solutions of (3.11) *by finding a fundamental set of solutions that consists of real functions. In this case, we use the rule above for a real root r of multiplicity m. If there is a non-real root $r = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ of multiplicity m, it's conjugate must also be a root of the same multiplicity. The pair of conjugate roots of multiplicity m contributes the $2m$ functions*

$$\begin{gathered} e^{\alpha t}\cos(\beta t), te^{\alpha t}\cos(\beta t), \dots, t^{m-1}e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t), te^{\alpha t}\sin(\beta t), \dots, t^{m-1}e^{\alpha t}\sin(\beta t) \end{gathered}$$

to the fundamental set of real-valued solutions.

With this preparation, we can state Leonard's Algorithm.

Theorem 3.7. [Leonard's Algorithm] *Let A be an $n \times n$ matrix, with characteristic polynomial $\chi_A(\lambda) = \det(A - \lambda I)$. Let $r_1, \dots, r_n$ be the solutions of the scalar, linear,* *homogeneous differential equation*

$$\chi_A(D)r(t) = 0$$

that satisfy the initial conditions

$$\begin{cases} r_1(0) = 1 \\ r_1'(0) = 0 \\ r_1''(0) = 0 \\ \vdots \\ r_1^{(n-1)}(0) = 0 \end{cases}, \qquad \begin{cases} r_2(0) = 0 \\ r_2'(0) = 1 \\ r_2''(0) = 0 \\ \vdots \\ r_2^{(n-1)}(0) = 0 \end{cases}, \qquad \dots \qquad , \begin{cases} r_n(0) = 0 \\ r_n'(0) = 0 \\ r_n''(0) = 0 \\ \vdots \\ r_n^{(n-1)}(0) = 1 \end{cases}.$$

Then,

$$e^{At} = r_1(t)I + r_2(t)A + r_3(t)A^2 + \cdots + r_n(t)A^{n-1}.$$

We'll first prove that this algorithm works, and then discuss the problem of how to find the solutions $r_1, \dots, r_n$, which we will call the **principal solutions** of $\chi_A(D)r = 0$.

The first step in the proof of Leonard's Algorithm is the following lemma.

Lemma 3.8. *Let A be an $n \times n$ matrix, with characteristic polynomial*

$$\chi_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

Then the unique matrix valued function $\Phi(t)$ which satisfies the differential equation

$$\chi_A(D)\Phi(t) = c_n\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) + \cdots + c_1\Phi'(t) + c_0\Phi(t) = 0 \tag{3.12}$$

with the initial conditions

$$\Phi(0) = I, \quad \Phi'(0) = A, \quad \Phi''(0) = A^2, \quad \dots \quad , \Phi^{(n-1)}(0) = A^{n-1} \tag{3.13}$$

is $\Phi(t) = e^{tA}$.

Proof. Let $\Phi(t) = [\varphi_{ij}(t)]$. Since the c_k's are scalars, each entry of $\Phi(t)$ satisfies the differential equation $\chi_A(D)\varphi_{ij}(t) = 0$, which initial conditions $\varphi_{ij}^{(k)}(0) =$ ij-th entry of A^{k-1} for $k = 0, \dots, n-1$. Thus, the function $\Phi(t)$ is unique, by the existence and uniqueness of solutions for scalar differential equations.

Next, we check that $\Phi(t) = e^{tA}$ is a solution. In this case, we know that $\Phi^{(k)}(t) = A^k e^{tA}$, so certainly the initial conditions are satisfied. Plugging $\Phi(t) = e^{tA}$ into the differential equation gives

$$\begin{aligned} \chi_A(D)\Phi(t) &= c_n\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) + \cdots + c_1\Phi'(t) + c_0\Phi(t) \\ &= c_nA^ne^{At} + c_{n-1}A^{n-1}e^{At} + \cdots + c_1Ae^{At} + c_0e^{At} \\ &= (c_nA^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I)e^{At} \\ &= \chi_A(A)e^{At}. \end{aligned}$$

But, $\chi_A(A) = 0$ by the Cayley-Hamilton Theorem. Thus, $\chi_A(D)\Phi(t) = 0$, so $\Phi(t) = e^{At}$ is a solution of this initial value problem.

$\square$

In order to complete the proof the Leonard's algorithm works, it will suffice to show that the function

$$\Phi(t) = \sum_{j=1}^{n-1} r_{j+1}(t) A^j$$

satisfies the differential equation (3.12) and the initial conditions (3.13). But this is almost trivial.

We have

$$\begin{aligned}
\chi_A(D)\Phi(t) &= \chi_A(D)\left[\sum_{j=1}^{n-1} r_{j+1}(t)A^j\right] \\
&= \sum_{j=0}^{n-1} [\chi_A(D) r_{j+1}(t)] A^j \\
&= \sum_{j=0}^{n-1} 0A^j && \text{since } \chi_A(D) r_{j+1} = 0 \\
&= 0.
\end{aligned}$$

It's easy to see that $\Phi(t)$ satisfies the initial conditions (3.13) because of the initial conditions imposed on the $r_j(t)$'s. Thus, by the uniqueness part of the lemma, we must have $\Phi(t) = e^{At}$.

In order to make this algorithm practical, we need to find the principal solutions. To approach this problem, let $f_1, \ldots, f_n$ be a fundamental set of solutions for $\chi_A(D)r = 0$. Thus, every solution of $\chi_A(D)r = 0$ can be written as $r = c_1 f_1 + c_2 f_2 + \ldots c_n f_n$ for some constants $c_1, \ldots, c_n$. Let

$$\mathcal{F} = \begin{bmatrix} f_1 & f_2 & \ldots & f_n \end{bmatrix},$$

which is an ordered basis for the n-dimensional vector space of solutions of $\chi_A(D)r = 0$. Then every solution can be written as $\mathcal{F}c$, where $c = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T$ is a constant vector.

Suppose that we want to find the solution of $\chi_A(D)r = 0$ that satisfies the initial conditions

$$\left\{\begin{aligned}
r(0) &= \gamma_1 \\
r'(0) &= \gamma_2 \\
&\vdots \\
r^{(n-1)}(0) &= \gamma_n,
\end{aligned}\right. \tag{3.14}$$

for some given constants $\gamma_1, \ldots, \gamma_n$.

We can write $r = c_1 f_1 + \cdots + c_n f_n$, so we need to determine the c_j's so that the initial conditions are satisfied. We have $r^{(k)} = c_1 f_1^{(k)} + \cdots + c_n f_n^{(k)}$, so the initial conditions (3.14) become

$$\begin{aligned}
c_1 f_1(0) + c_2 f_2(0) + \cdots + c_n f_n(0) &= \gamma_1 \\
c_1 f_1'(0) + c_2 f_2'(0) + \cdots + c_n f_n'(0) &= \gamma_2 \\
c_1 f_1''(0) + c_2 f_2''(0) + \cdots + c_n f_n''(0) &= \gamma_3 \\
&\vdots \\
c_1 f_1^{(n-1)}(0) + c_2 f_2^{(n-1)}(0) + \cdots + c_n f_n^{(n-1)}(0) &= \gamma_n
\end{aligned}$$

We can write this system of equations in matrix form as

$$M_0 c = \gamma$$

where

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \qquad \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix},$$

and

$$M_0 = \begin{bmatrix} f_1(0) & f_2(0) & \dots & f_n(0) \\ f_1'(0) & f_2'(0) & \dots & f_n'(0) \\ f_1''(0) & f_2''(0) & \dots & f_n''(0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(0) & f_2^{(n-1)}(0) & \dots & f_n^{(n-1)}(0) \end{bmatrix}$$

is the Wronskian matrix of the solutions in $\mathcal{F}$, evaluated at zero. Since the Wronskian matrix is invertible, the solution of the system $M_0 c = \gamma$ is $c = M_0^{-1}\gamma$. The solution r of the differential equation $\chi_A(D)r = 0$ that satisfies the initial conditions (3.14) is then $r = \mathcal{F}c = \mathcal{F}M_0^{-1}\gamma$.

Now suppose that we want to find solutions $r_1, \dots, r_n$ corresponding to vectors $\gamma_1, \dots, \gamma_n$ of initial conditions. We then have $r_j = \mathcal{F}M_0^{-1}\gamma_j$. If we write

$$\mathcal{R} = \begin{bmatrix} r_1 & r_2 & \dots & r_n \end{bmatrix}$$

for the row of the r_j's and let

$$\Gamma = [\gamma_1 \mid \gamma_2 \mid \cdots \mid \gamma_n]$$

be the matrix whose columns are the γ_j's, we can put the equations $r_j = \mathcal{F}M_0^{-1}\gamma_j$ together into the single equation

$$\mathcal{R} = \mathcal{F}M_0^{-1}\Gamma.$$

If we want $\mathcal{R}$ to be the row of the principal solutions, the corresponding matrix of initial conditions is just $\Gamma = I$. Thus, we can find the principal solutions as

$$\mathcal{R} = \mathcal{F}M_0^{-1}.$$

We can now give some examples of finding e^{At} by Leonard's algorithm.

Example 3.9. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & -4 \\ 3 & 5 & -7 \\ 1 & 2 & -2 \end{bmatrix}$$

The characteristic polynomial is

$$\chi_A(\lambda) = \lambda^3 - 6\,\lambda^2 + 11\,\lambda - 6 = (\lambda - 1)\,(\lambda - 2)\,(\lambda - 3)$$

Since there are three distinct eigenvalues, the matrix A must be diagonalizable. We leave it as an exercise for the reader to compute e^{tA} by the diagonalization method and check the result is the same as we get here by Leonard's algorithm. For a fundamental set of solutions of the equation $\chi_A(D)r = 0$, we can take

$$\mathcal{F} = \begin{bmatrix} e^t & e^{2t} & e^{3t} \end{bmatrix}.$$

The Wronskian matrix of $\mathcal{F}$ is

$$M(t) = \begin{bmatrix} e^t & e^{2\,t} & e^{3\,t} \\ e^t & 2\,e^{2\,t} & 3\,e^{3\,t} \\ e^t & 4\,e^{2\,t} & 9\,e^{3\,t} \end{bmatrix}$$

The value of the Wronskian matrix at 0 is

$$M_0 = M(0) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

and a calculator computations shows

$$M_0^{-1} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}.$$

We can then find the row $\mathcal{R}$ of principal solutions by

$$\begin{aligned} \mathcal{R} &= \mathcal{F} M_0^{-1} \\ &= \begin{bmatrix} e^t & e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 3\,e^t - 3\,e^{2\,t} + e^{3\,t} & -5/2\,e^t + 4\,e^{2\,t} - 3/2\,e^{3\,t} & 1/2\,e^t - e^{2\,t} + 1/2\,e^{3\,t} \end{bmatrix}. \end{aligned}$$

Then, by Leonard's Algorithm, we have

$$\begin{aligned} e^{At} &= r_1(t)I + r_2(t)A + r_3(t)A^2 \\ &= \left(3\,e^t - 3\,e^{2\,t} + e^{3\,t}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(-5/2\,e^t + 4\,e^{2\,t} - 3/2\,e^{3\,t}\right) \begin{bmatrix} 3 & 2 & -4 \\ 3 & 5 & -7 \\ 1 & 2 & -2 \end{bmatrix} \\ &\quad + \left(1/2\,e^t - e^{2\,t} + 1/2\,e^{3\,t}\right) \begin{bmatrix} 11 & 8 & -18 \\ 17 & 17 & -33 \\ 7 & 8 & -14 \end{bmatrix} \\ &= \begin{bmatrix} e^t - 2\,e^{2\,t} + 2\,e^{3\,t} & -e^t + e^{3\,t} & e^t + 2\,e^{2\,t} - 3\,e^{3\,t} \\ e^t - 5\,e^{2\,t} + 4\,e^{3\,t} & -e^t + 2\,e^{3\,t} & e^t + 5\,e^{2\,t} - 6\,e^{3\,t} \\ e^t - 3\,e^{2\,t} + 2\,e^{3\,t} & -e^t + e^{3\,t} & e^t + 3\,e^{2\,t} - 3\,e^{3\,t} \end{bmatrix}. \end{aligned}$$

Suppose that we want to solve the vector differential equation

$$\frac{dy}{dt} = Ay$$

subject to the initial condition

$$y(0) = y_0 = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}.$$

The know the solution is $y = e^{At}y_0$, so we have

$$\begin{aligned} y(t) &= \begin{bmatrix} e^t - 2\,e^{2\,t} + 2\,e^{3\,t} & -e^t + e^{3\,t} & e^t + 2\,e^{2\,t} - 3\,e^{3\,t} \\ e^t - 5\,e^{2\,t} + 4\,e^{3\,t} & -e^t + 2\,e^{3\,t} & e^t + 5\,e^{2\,t} - 6\,e^{3\,t} \\ e^t - 3\,e^{2\,t} + 2\,e^{3\,t} & -e^t + e^{3\,t} & e^t + 3\,e^{2\,t} - 3\,e^{3\,t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -8\,e^t - 8\,e^{2\,t} + 15\,e^{3\,t} \\ -8\,e^t - 20\,e^{2\,t} + 30\,e^{3\,t} \\ -8\,e^t - 12\,e^{2\,t} + 15\,e^{3\,t} \end{bmatrix}. \end{aligned}$$

Example 3.10. Consider the matrix

$$A = \begin{bmatrix} -16 & 6 & -5 \\ -36 & 14 & -10 \\ 18 & -6 & 7 \end{bmatrix}.$$

We want to find e^{At}. The characteristic polynomial is

$$\chi_A(\lambda) = \lambda^3 - 5\,\lambda^2 + 8\,\lambda - 4 = (\lambda - 1)\,(\lambda - 2)^2.$$

Since we have a multiple root, we can't tell without further calculation if A is diagonalizable. We leave it as an exercise for the reader to show that A is in fact diagonalizable and to find e^{At} by the diagonalization method. We'll use Leonard's algorithm. A fundamental set of solutions for the equation $\chi_A(D)r = 0$ is

$$\mathcal{F} = \begin{bmatrix} e^t & e^{2t} & te^{2t} \end{bmatrix}.$$

The Wronskian matrix of $\mathcal{F}$ is

$$M(t) = \begin{bmatrix} e^t & e^{2\,t} & te^{2\,t} \\ e^t & 2\,e^{2\,t} & e^{2\,t} + 2\,te^{2\,t} \\ e^t & 4\,e^{2\,t} & 4\,e^{2\,t} + 4\,te^{2\,t} \end{bmatrix}$$

and the value of the Wronskian matrix at $t = 0$ is

$$M_0 = M(0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{bmatrix}$$

and the inverse of this is

$$M_0^{-1} = \begin{bmatrix} 4 & -4 & 1 \\ -3 & 4 & -1 \\ 2 & -3 & 1 \end{bmatrix}.$$

Thus, the row $\mathcal{R}$ of principal solutions is

$$\begin{aligned}
\mathcal{R} &= \mathcal{F}M_0^{-1} \\
&= \begin{bmatrix} e^t & e^{2t} & te^{2t} \end{bmatrix} \begin{bmatrix} 4 & -4 & 1 \\ -3 & 4 & -1 \\ 2 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4\,e^t - 3\,e^{2\,t} + 2\,te^{2\,t} & -4\,e^t + 4\,e^{2\,t} - 3\,te^{2\,t} & e^t - e^{2\,t} + te^{2\,t} \end{bmatrix}.
\end{aligned}$$

Thus, by Leonard's Algorithm, we have

$$\begin{aligned}
e^{At} &= r_1(t)I + r_2(t)A + r_3(t)A^2 \\
&= \left(4\,e^t - 3\,e^{2\,t} + 2\,te^{2\,t}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(-4\,e^t + 4\,e^{2\,t} - 3\,te^{2\,t}\right) \begin{bmatrix} -16 & 6 & -5 \\ -36 & 14 & -10 \\ 18 & -6 & 7 \end{bmatrix} \\
&\quad + \left(e^t - e^{2\,t} + te^{2\,t}\right) \begin{bmatrix} -50 & 18 & -15 \\ -108 & 40 & -30 \\ 54 & -18 & 19 \end{bmatrix} \\
&= \begin{bmatrix} 18\,e^t - 17\,e^{2\,t} & -6\,e^t + 6\,e^{2\,t} & 5\,e^t - 5\,e^{2\,t} \\ 36\,e^t - 36\,e^{2\,t} & -12\,e^t + 13\,e^{2\,t} & 10\,e^t - 10\,e^{2\,t} \\ -18\,e^t + 18\,e^{2\,t} & 6\,e^t - 6\,e^{2\,t} & -5\,e^t + 6\,e^{2\,t} \end{bmatrix}.
\end{aligned}$$

We would predict from the diagonalization method that the entries of e^{At} should be linear combinations of e^t and e^{2t}, and so it proves, even though Leonard's algorithm involves the additional function te^{2t}.

Next, consider the matrix

$$B = \begin{bmatrix} 64 & -21 & 17 \\ 96 & -31 & 26 \\ -108 & 36 & -28 \end{bmatrix}.$$

This matrix has the same characteristic polynomial as A, but the reader can check that B is not diagonalizable. Since the principal solutions depend only on the characteristic polynomial, we use the same principal solutions as for A. Thus, we calculate

$$\begin{aligned}
e^{Bt} &= r_1(t)I + r_2(t)B + r_3(t)B^2 \\
&= \left(4\,e^t - 3\,e^{2\,t} + 2\,te^{2\,t}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(-4\,e^t + 4\,e^{2\,t} - 3\,te^{2\,t}\right) \begin{bmatrix} 64 & -21 & 17 \\ 96 & -31 & 26 \\ -108 & 36 & -28 \end{bmatrix} \\
&\quad + \left(e^t - e^{2\,t} + te^{2\,t}\right) \begin{bmatrix} 244 & -81 & 66 \\ 360 & -119 & 98 \\ -432 & 144 & -116 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} -8\,e^{t}+9\,e^{2\,t}+54\,te^{2\,t} & 3\,e^{t}-3\,e^{2\,t}-18\,te^{2\,t} & -2\,e^{t}+2\,e^{2\,t}+15\,te^{2\,t} \\ -24\,e^{t}+24\,e^{2\,t}+72\,te^{2\,t} & 9\,e^{t}-8\,e^{2\,t}-24\,te^{2\,t} & -6\,e^{t}+6\,e^{2\,t}+20\,te^{2\,t} \\ -108\,te^{2\,t} & 36\,te^{2\,t} & e^{2\,t}-30\,te^{2\,t} \end{bmatrix}.$$

This time the function te^{2t} does appear in the final answer.

Example 3.11. In this example, we find e^{At} for the matrix

$$A = \begin{bmatrix} -129 & 44 & -36 \\ -204 & 71 & -56 \\ 222 & -74 & 63 \end{bmatrix}.$$

The characteristic polynomial is

$$\chi_A(\lambda) = \lambda^3 - 5\,\lambda^2 + 11\,\lambda - 15 = (\lambda - 3)\left(\lambda^2 - 2\,\lambda + 5\right).$$

Using the quadratic formula, we see that the roots of the characteristic polynomial are 3 and $1 \pm 2i$. Since A has real entries, e^{At} must have real entries, so it makes sense to use real solutions of the equation $\chi_A(D)r = 0$. Thus, in this case we can take a fundamental set of solutions to be

$$\mathcal{F} = \begin{bmatrix} e^{3t} & e^t\cos(2t) & e^t\sin(2t) \end{bmatrix}.$$

We leave it to the reader to check that the Wronskian matrix at 0 is

$$M_0 = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ 9 & -3 & 4 \end{bmatrix}$$

and that the row of principal solutions is

(3.15)

$$\mathcal{R} = \big[\; 5/8\,e^{3\,t}+3/8\,e^{t}\cos(2\,t)-\tfrac{9}{8}\,e^{t}\sin(2\,t) \quad -1/4\,e^{3\,t}+1/4\,e^{t}\cos(2\,t)+3/4\,e^{t}\sin(2\,t) \quad 1/8\,e^{3\,t}-1/8\,e^{t}\cos(2\,t)-1/8\,e^{t}\sin(2\,t)\;\big].$$

Thus, we have

$$\begin{aligned} e^{At} &= r_1(t)I + r_2(t)A + r_3(t)A^2 \\ &= \left(5/8\,e^{3\,t}+3/8\,e^{t}\cos(2\,t)-\frac{9}{8}\,e^{t}\sin(2\,t)\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad + \left(-1/4\,e^{3\,t}+1/4\,e^{t}\cos(2\,t)+3/4\,e^{t}\sin(2\,t)\right)\begin{bmatrix} -129 & 44 & -36 \\ -204 & 71 & -56 \\ 222 & -74 & 63 \end{bmatrix} \\ &\quad + \left(1/8\,e^{3\,t}-1/8\,e^{t}\cos(2\,t)-1/8\,e^{t}\sin(2\,t)\right)\begin{bmatrix} -327 & 112 & -88 \\ -600 & 209 & -160 \\ 444 & -148 & 121 \end{bmatrix} \\ &= \begin{bmatrix} -8\,e^{3\,t}+9\,e^{t}\cos(2\,t)-57\,e^{t}\sin(2\,t) & 3\,e^{3\,t}-3\,e^{t}\cos(2\,t)+19\,e^{t}\sin(2\,t) & -2\,e^{3\,t}+2\,e^{t}\cos(2\,t)-16\,e^{t}\sin(2\,t) \\ -24\,e^{3\,t}+24\,e^{t}\cos(2\,t)-78\,e^{t}\sin(2\,t) & 9\,e^{3\,t}-8\,e^{t}\cos(2\,t)+26\,e^{t}\sin(2\,t) & -6\,e^{3\,t}+6\,e^{t}\cos(2\,t)-22\,e^{t}\sin(2\,t) \\ 111\,e^{t}\sin(2\,t) & -37\,e^{t}\sin(2\,t) & e^{t}\cos(2\,t)+31\,e^{t}\sin(2\,t) \end{bmatrix}. \end{aligned}$$

For the final topic in this set of notes, we discuss linear inhomogeneous systems of equations with constant coefficients. Thus, we want to consider the system

$$\frac{dy}{dt} = Ay + b(t) \tag{3.16}$$

with the initial condition $y = y_0$, where A is an $n \times n$ matrix and b is a given function with values in $\mathbb{K}^n$. The formula for the solution is a special case of the variation of parameters formula (2.14). In this case, we can give an alternative derivation using an integrating factor. Write the differential equation (3.16) as

$$\frac{dy}{dt} - Ay = b(t).$$

Multiply both sides of this equation on the left by e^{-tA} to get

$$e^{-tA}\frac{dy}{dt} - e^{-tA}y = e^{-tA}b(t).$$

The left-hand side of this equation is

$$\frac{d}{dt}\Big[e^{-tA}y\Big],$$

so the equation becomes

$$\frac{d}{dt}\Big[e^{-tA}y\Big] = e^{-tA}b(t).$$

Integrating both sides from 0 to t gives

$$e^{-tA}y = c + \int_0^t e^{-sA}b(s)\,ds \tag{3.17}$$

where c is a constant. Plugging in $t = 0$ shows that $c = y_0$. If we plug this into equation (3.17) and solve for y, we get

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}b(s)\,ds,$$

which is the variation of parameters formula in this case.

Exercise 3.12. Solve the differential equation

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} t \\ e^t \end{bmatrix}$$

subject to the initial condition

$$y(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

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