Problem 1.

The following matrix A is diagonalizable. Find e^{At} by the diagonalization method.

$$A = \begin{bmatrix} 7 & -9 \\ 6 & -8 \end{bmatrix}$$

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40 pts.

Problem 2. Show that the following matrix N is nilpotent. Use this fact to calculate e^{Nt}.

$$N = \begin{bmatrix} 111 & -46 & -106 & 11 \\ -43 & 21 & 52 & 11 \\ 127 & -54 & -126 & 6 \\ -88 & 37 & 86 & -6 \end{bmatrix}$$

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60 pts.

Problem 3. Consider the matrix

$$A = \begin{bmatrix} 74 & -53 & -6 & -64 & -20 & -30 \\ 61 & -43 & -6 & -54 & -17 & -27 \\ 117 & -95 & -1 & -101 & -22 & -39 \\ 43 & -35 & 0 & -36 & -8 & -13 \\ 29 & -18 & -6 & -26 & -10 & -17 \\ -63 & 54 & 0 & 55 & 9 & 22 \end{bmatrix}$$

A. Find a basis of each of the generalized eigenspaces of A.

B. Find the matrices S and N so that $A = S + N$, S is diagonalizable, N is nilpotent, and $SN = NS$. Verify that your answers for S and N really satisfy these conditions. (It is not necessary to go all the way to the Jordan Canonical Form of A.)

C. Use S and N to compute $e^{At} = e^{tS}e^{tN}$.

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80 pts. **Problem 4**. In each part, use Leonard's Algorithm to find e^{At}.

A.

$$A = \begin{bmatrix} 27 & -40 & -46 \\ -8 & 13 & 14 \\ 22 & -34 & -38 \end{bmatrix}$$

B.

$$A = \begin{bmatrix} -14 & -2 & -10 \\ -40 & -3 & -25 \\ 32 & 4 & 22 \end{bmatrix}$$

C.

$$A = \begin{bmatrix} -385 & 1010 \\ -146 & 383 \end{bmatrix}$$

D.

$$A = \begin{bmatrix} -51 & -15 & 10 & 12 \\ -1784 & 33 & -188 & 432 \\ -1900 & 36 & -201 & 460 \\ -894 & -49 & -31 & 215 \end{bmatrix}$$

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40 pts. **Problem 5**. Recall that we have shown

$$e^{tA}e^{tB} = e^{tB}e^{tA}$$

provided that A and B commute. To see what happens if A and B do not commute, consider the function

$$\Psi(t) = e^{tA}e^{tB}e^{-tA}e^{-tB}$$

so if A and B commute, $\Psi(t) = I$ for all t. To see how much Ψ differs from the identity in the general case, find the first two nonzero terms in the Taylor expansion of $\Psi(t)$, centered at $t = 0$.

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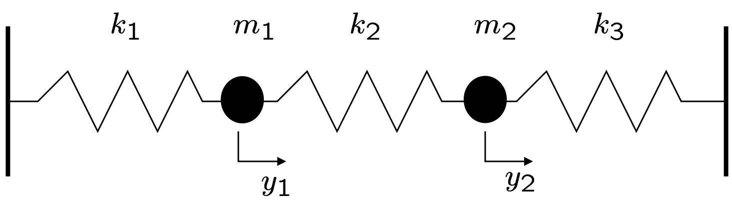
60 pts. **Problem 6**. Consider the following mechanical system.



We have two weights of masses m_1 and m_2 connected by three springs to a stationary frame. The spring constants of the three springs are k_1, k_2 and k_3, as indicated in the diagram. The weights are constrained to move in a horizontal line. The deviation of the two weights from their equilibrium positions are y_1 and y_2 respectively, measured positive to the right.

We first consider the case where there is no friction.

The eqution of motion of the first mass is

$$m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$

and the equation of motion for the second weight is

$$m_2 y_2'' = -k_2(y_2 - y_1) - k_3 y_2.$$

To reduce this to a first order system we introduce new dependent variables z_1 and z_2 and add the equations $y_1' = z_1$ and $y_2' = z_2$. Thus, our system of equations becomes

$$\begin{aligned} y_1' &= z_1 \\ y_2' &= z_2 \\ z_1' &= -\frac{k_1 + k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2 \\ z_2' &= \frac{k_2}{m_2} y_1 - \frac{k_2 + k_3}{m_2} y_2. \end{aligned}$$

A. Write the system of equations above in matrix form.

B. Assume the values

$$m_1 = 1, \quad m_2 = 2, \quad k_1 = 1, \quad k_2 = 2, \quad k_3 = 2.$$

Find e^{At}, where A is the coefficient matrix of the system, using one of the methods developed in class.

Find the solution for the initial conditions

$$y_1(0) = 1, \quad y_2(0) = -1, \quad y_1'(0) = 0, \quad y_2'(0) = 0.$$

Provide a graph of y_1 and y_2 as functions of time.

C. Assume now that the weights are subject to frictional forces proportional to their velocities, i.e., the force on the first weight due to friction is $-b_1 y_1'$ and the force on the second weight due to friction is $-b_2 y_2'$, where b_1 and b_2 are positive constants. Find the new system of equations $dy/dt = By$.

D. Assume the values

$$m_1 = 1, \quad m_2 = 1$$
$$k_1 = 1, \quad k_2 = 1, \quad k_3 = 1$$
$$b_1 = 1/2, \quad b_2 = 1/2.$$

Find e^{Bt}, using one of the methods developed in class. Solve the system subject to the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_1'(0) = 0 \quad y_2'(0) = 0.$$

Provide a graph showing y_1 and y_2 as functions of time.

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60 pts.

Problem 7. In each part, consider the system

$$\frac{dy}{dt} = Ay.$$

Determine if the fixed point at $y = 0$ is asymptotically stable, stable but not asymptotically stable, or unstable. Justify your answers, of course.

A.

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$$

B.

$$A = \begin{bmatrix} 18 & -20 \\ 15 & -17 \end{bmatrix}$$

C.

$$A = \begin{bmatrix} -717 & 1885 & -2903 & -1026 & 491 \\ 110 & -212 & 418 & 184 & -94 \\ 278 & -608 & 1082 & 440 & -220 \\ -185 & 63 & -599 & -408 & 227 \\ -214 & 104 & -704 & -460 & 254 \end{bmatrix}$$

D.

$$A = \begin{bmatrix} -27 & 1363 & -572 & 460 & -336 \\ 44 & -162 & 195 & 42 & -15 \\ 77 & -456 & 403 & 7 & 21 \\ -245 & 109 & -802 & -536 & 298 \\ -271 & 148 & -897 & -581 & 321 \end{bmatrix}$$

EXAM

Exam #3

Math 3351, Spring 2003

April 14, 2003

- This is a **Take-Home Exam**, due April 23.
- Write all of your answers on separate sheets of paper. You can keep the exam questions.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **340 points total**.

Good luck!