EXAM

Exam # 2
Take-home Exam

Math 3351, Spring 2003

Feb. 28, 2003

ANSWERS

70 pts. **Problem 1**. Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -12 & -4 & 1 & 11 \\ 1 & 1 & -1 & 0 & 0 & 5 \\ 4 & 3 & -6 & -1 & 1 & 17 \\ 2 & -1 & -8 & -3 & 1 & 3 \\ 2 & 1 & -4 & -1 & 2 & 6 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A. Find a basis for the nullspace of A.

Answer:

The nullspace is the space of solutions of the system $A\mathbf{x} = \mathbf{0}$. Since R is row equivalent to A, the system $R\mathbf{x} = \mathbf{0}$ has the same solutions. So, we read off the solutions from R, using zero as the right-hand side of the system.

Looking at R, call the variables x_1 through x_6. Then x_1, x_2 and x_5 are leading variables and x_3, x_4 and x_6 are free variables, say

$$\begin{aligned} x_3 &= \alpha \\ x_4 &= \beta \\ x_6 &= \gamma. \end{aligned}$$

Reading the rows of R from the bottom up gives the equations

$$\begin{aligned} x_5 - x_6 = 0 &\implies x_5 = x_6 = \gamma \\ x_2 + 2x_3 + x_4 + 2x_6 = 0 &\implies x_2 = -2x_3 - x_4 - 2x_6 = -2\alpha - \beta - 2\gamma \\ x_1 - 3x_3 - x_4 + 3x_6 = 0 &\implies x_1 = 3x_3 + x_4 - 3x_6 = 3\alpha + \beta - 3\gamma. \end{aligned}$$

Putting these equations together gives the family of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3\alpha + \beta - 3\gamma \\ 2\alpha - \beta - 2\gamma \\ \alpha \\ \beta \\ \gamma \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, as we discussed, the vectors

$$\begin{bmatrix}3\\2\\1\\0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1\\-1\\0\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-3\\-2\\0\\0\\1\\1\end{bmatrix}$$

form a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

Answer:

A basis for the rowspcae is given by the nonzero rows in the RREF of A. Thus, the vectors

$$\begin{bmatrix}1 & 0 & -3 & -1 & 0 & 3\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 2 & 1 & 0 & 2\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 0 & 1 & -1\end{bmatrix}$$

form a basis of the rowspace of A.

C. Find a basis for the columnspace of A

Answer:

To find a basis for the columnspace of A, we find the columns in R that contain the leading entries (columns 1,2 and 5) and take *the corresponding columuns from the original matrix* A. Thus, the vectors

$$\begin{bmatrix}4\\1\\4\\2\\2\end{bmatrix}, \quad \begin{bmatrix}0\\1\\3\\-1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix}$$

form a basis for the columnspace of A.

D. What is the rank of A?

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50 pts.

Problem 2. Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix}4\\1\\2\\1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\2\\0\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}3\\2\\1\\1\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}6\\8\\0\\3\end{bmatrix}.$$

A. Cut down the list of vectors above to a basis for S. What is the dimension of S?

Answer:

We form the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \mid \mathbf{w}_1 \mid \mathbf{w}_2 \end{bmatrix} = \left[\begin{array}{cccc|cc} 4 & 2 & 3 & 6 & 18 & 15 \\ 1 & 2 & 2 & 8 & 12 & 2 \\ 2 & 0 & 1 & 0 & 5 & 9 \\ 1 & 1 & 1 & 3 & 7 & 3 \end{array}\right].$$

The RREF of A is

$$R = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

In the part of R to the right of the bar, we see that the first three columns of R are independent and that the fourth column is a linear combination of the first three. The same must be true for A, we we conclude that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for S. Since S has a basis with three elements, the dimension of S is 3.

B. For each of the following vectors, determine if the vector is in S and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} 18 \\ 12 \\ 5 \\ 7 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 15 \\ 2 \\ 9 \\ 3 \end{bmatrix}$$

Answer:

In the matrix R above, we see that the 5th column is not a linear combination of the first 4 columns (because of the last row), so $\mathbf{w}_1$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, so $\mathbf{w}_1 \notin S$.

From the matrix R above, we see that $\operatorname{col}_6(R) = 4\operatorname{col}_1(R) - 2\operatorname{col}_2(R) + \operatorname{col}_3(R)$. The same must be true about the columns of A, so we have

$$\mathbf{w}_2 = 4\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3,$$

and so $\mathbf{w}_2 \in S$.

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50 pts. **Problem 3**. Let A be a 6×8 matrix and let B be a 7×7 matrix.

A. What is the largest possible value of the rank of A?

Answer: 6.

B. If the nullspace of A has dimension 5, what is the rank of A?

Answer:

The rank theorem says the rank plus the nullity of A must equal the number of columns. The number of columns is 8, and we're given that the nullity is 5, so we must have $\text{rank}(A) = 3$

C. If the rowspace of B has dimension 4, what is the dimension of the nullspace of B?

Answer:

The rank is equal to the dimension of the rowspace or columnspace. Thus, we know $\text{rank}(B) = 4$. The rank plus the nullity of B is equal to 7 (the number of columns). Thus, the nullity of B is 3. By definition, the nullity is the dimension of the nullspace, so the nullspace of B has dimension 3.

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40 pts.

Problem 4. Let

$$A = \begin{bmatrix} -4 & 4 \\ -3 & 4 \end{bmatrix}.$$

Find the characteristic polynomial of A and the eigenvalues of A.

Answer:
The characteristic polynomial $p(\lambda)$ is given by

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} -4-\lambda & 4 \\ -3 & 4-\lambda \end{vmatrix} \\ &= (-4-\lambda)(4-\lambda) - (-3)(4) \\ &= -16 + 4\lambda - 4\lambda + \lambda^2 + 12 \\ &= \lambda^2 - 4. \end{aligned}$$

The characteristic polynomials factors as $\lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$, so the roots (eigenvalues) are 2 and -2.

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60 pts.

Problem 5. In each part you are given an matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A. Determine if A is diagonalizable, and if it is, find a matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

A.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \qquad \text{Eigenvalues} = 1, 2.$$

Answer:

For the eigenvalue 1, we have

$$A - \lambda I = A - I = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

The reduced row echelon form of $A - I$ is

$$R_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We want to find the nullspace of $A - I$, which is the same as the nullspace of R_1. Call the variables x, y, z. Then z is a free variable, say $z = \alpha$. The second row of R_1 gives us the equation $y - z = 0$, so we have $y = \alpha$. The first row of R_1 gives us $x - 2z = 0$, so $x = 2\alpha$. Thus, the nullspace of $A - I$ is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace for $\lambda = 1$ is one dimensional with basis vector

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Now consider the eigenvalue $\lambda = 2$. We have

$$A - \lambda I = A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

The RREF of $A - 2I$ is

$$R_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Once again z is a free variable, say $z = \alpha$ The second row gives $y - z = 0$, so $y = \alpha$. The top row of R_2 gives $x - z = 0$ and so $x = \alpha$. Thus, the nullspace of $A - 2I$ is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace for eigenvalue $\lambda = 2$ is one dimensional with basis vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since we've only found two independent eigenvectors, not three, we conclude that A **is not diagonalizable**.

B.

$$A = \begin{bmatrix} 8 & 9 & 9 \\ 0 & 2 & 0 \\ -6 & -9 & -7 \end{bmatrix}, \qquad \text{Eigenvalues} = -1, 2.$$

Answer:

For the eigenvalue $\lambda = -1$ we have

$$A - \lambda I = A + I = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 3 & 0 \\ -6 & -9 & -6 \end{bmatrix}.$$

The RREF of $A + I$ is

$$R_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Again, z is a free variable, say $z = \alpha$. Then second row gives $y = 0$. The first row gives $x + z = 0$, so $x = -\alpha$. Thus, the nullspace of $A + I$ is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace for $\lambda = -1$ is one dimensional with basis vector

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Next, we consider the eigenvalue $\lambda = 2$. In this case we have

$$A - \lambda I = A - 2I = \begin{bmatrix} 6 & 9 & 9 \\ 0 & 0 & 0 \\ -6 & -9 & -9 \end{bmatrix}.$$

The RREF of $A - 2I$ is

$$R_2 = \begin{bmatrix} 1 & 3/2 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that y and z are free variables, say $y = \alpha$ and $z = \beta$. The top row of R_2 gives us the equation

$$x + \frac{3}{2}y + \frac{3}{2}z = 0,$$

so we have

$$x = -\frac{3}{2}\alpha - \frac{3}{2}\beta.$$

Thus, the nullspace of $A - 2I$ is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\alpha - \frac{3}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the eigenspace for $\lambda = 2$ is two dimensional with basis

$$\begin{bmatrix} -3/2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}.$$

Since we've found three independent eigenvectors for the 3×3 matrix A, we conclude that A is diagonalizable. If we set

$$P = \begin{bmatrix} -1 & -3/2 & -3/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

then D is a diagonal matrix and $P^{-1}AP = D$.

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80 pts.

Problem 6.
Let $\mathcal{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$ be the ordered basis of $\mathbb{R}^2$ where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

Answer:

The defining equation of $S_{\mathcal{E}\mathcal{U}}$ is $\mathcal{U} = \mathcal{E}S_{\mathcal{E}\mathcal{U}}$ or, in matrix terms, $\text{mat}(\mathcal{U}) = IS_{\mathcal{E}\mathcal{U}}$, thus

$$S_{\mathcal{E}\mathcal{U}} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

We have $S_{\mathcal{U}\mathcal{E}} = (S_{\mathcal{E}\mathcal{U}})^{-1}$. Using a calculator to find the inverse, we get

$$S_{\mathcal{U}\mathcal{E}} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}.$$

B. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$L(\mathbf{u}_1) = 2\mathbf{u}_1 - 1\mathbf{u}_2$$
$$L(\mathbf{u}_2) = 3\mathbf{u}_1 - 5\mathbf{u}_2.$$

Find $[L]_{\mathcal{U}\mathcal{U}}$, the matrix of L with respect to the basis $\mathcal{U}$.

Answer:

The defining equation of $[L]_{\mathcal{U}\mathcal{U}}$ is

$$L(\mathcal{U}) = \mathcal{U}[L]_{\mathcal{U}\mathcal{U}}.$$

We are given the entries of $L(\mathcal{U}) = \begin{bmatrix} L(\mathbf{u}_1) & L(\mathbf{u}_2) \end{bmatrix}$, so we find the matrix that fills in the equation

$$\begin{bmatrix} 2\mathbf{u}_1 - 1\mathbf{u}_2 & 3\mathbf{u}_1 - 5\mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}.$$

We conclude that

$$[L]_{\mathcal{U}\mathcal{U}} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}.$$

C. Find the matrix of L with respect to the standard basis $\mathcal{E}$ of $\mathbb{R}^2$

Answer:

The change of coordinates formula for linear transformations is

$$[L]_{\mathcal{E}\mathcal{E}} = S_{\mathcal{E}\mathcal{U}}[L]_{\mathcal{U}\mathcal{U}}S_{\mathcal{U}\mathcal{E}},$$

so from our work above we have

$$[L]_{\mathcal{E}\mathcal{E}} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ -11 & 5 \end{bmatrix}$$

D. Let $\mathbf{v}$ be the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Answer:

By the change of coordinates formula for vectors, we have

$$\begin{aligned} [\mathbf{v}]_{\mathcal{U}} &= S_{\mathcal{U}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} \\ &= \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned}\mathbf{v} &= \mathcal{U}[\mathbf{v}]_{\mathcal{U}} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= -\mathbf{u}_1 + 3\mathbf{u}_2.\end{aligned}$$

b. Express $L(\mathbf{v})$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Answer:
Recall that we were given $L(\mathbf{u}_1)$ and $L(\mathbf{u}_2)$ above. Thus, we can compute

$$\begin{aligned}L(\mathbf{v}) &= L(-\mathbf{u}_1 + 3\mathbf{u}_2) \\ &= -L(\mathbf{u}_1) + 3L(\mathbf{u}_2) \\ &= -(2\mathbf{u}_1 - \mathbf{u}_2) + 3(3\mathbf{u}_1 - 5\mathbf{u}_2) \\ &= 7\mathbf{u}_1 - 14\mathbf{u}_2.\end{aligned}$$

c. Express $L(\mathbf{v})$ as a column vector.

Answer:
We know $L(\mathbf{v})$ as a combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ and we know $\mathbf{u}_1$ and $\mathbf{u}_2$, so we have

$$\begin{aligned}L(\mathbf{v}) &= 7\mathbf{u}_1 - 14\mathbf{u}_2 \\ &= 7\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 14\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ -7 \end{bmatrix}.\end{aligned}$$

d. Check the last part against $[L]_{\mathcal{E}\mathcal{E}}\mathbf{v}$.

Answer:
We should have

$$\begin{aligned}L(\mathbf{v}) &= [L(\mathbf{v})]_{\mathcal{E}} \\ &= [L]_{\mathcal{E}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} \\ &= [L]_{\mathcal{E}\mathcal{E}}\mathbf{v} \\ &= \begin{bmatrix} -8 & 2 \\ -11 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ -7 \end{bmatrix},\end{aligned}$$

which agrees with our result in the previous part.

40 pts. **Problem 7**. Let S be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 18 \\ 38 \\ 11 \\ 11 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 & 1 & -3 \\ -2 & 3 & -4 & 1 & -5 & 12 \\ -8 & 15 & -22 & 3 & -25 & 56 \\ 2 & -3 & 4 & -1 & 5 & -12 \\ -8 & 14 & -20 & 3 & -24 & 53 \\ 7 & 9 & -25 & -2 & 1 & 18 \end{bmatrix}.$$

Define K by

$$K = \{\mathbf{v} \in S \mid A\mathbf{v} = \mathbf{0}\}.$$

Find a basis of K. Explain your reasoning.

Answer:
A check with a calculator shows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are independent and so are a basis of S. Thus, S has dimension 4. Let $\mathcal{V}$ be the ordered basis

$$\mathcal{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$$

of S.

Let $T\colon S \to \mathbb{R}^6$ be the linear transformation induced by multiplication by A, i.e., $T(v) = Av$. Of course, then

$$K = \{\mathbf{s} \in S \mid T(\mathbf{s}) = 0\}.$$

We have the ordered basis $\mathcal{V}$ for S and the standard basis $\mathcal{E}$ of $\mathbb{R}^6$. Let $B = [T]_{\mathcal{E}\mathcal{V}}$ be the matrix of T with respect to these basis. The defining equation of B is

$$T(\mathcal{V}) = \mathcal{E}[T]_{\mathcal{E}\mathcal{V}} = \mathcal{E}B. \tag{1}$$

Of course

$$T(\mathcal{V}) = \begin{bmatrix} T(\mathbf{v}_1) & T(\mathbf{v}_2) & T(\mathbf{v}_3) & T(\mathbf{v}_4) \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 & A\mathbf{v}_4 \end{bmatrix}.$$

In matrix terms, equation (1) becomes $\mathrm{mat}(T(\mathcal{V})) = \mathrm{mat}(\mathcal{E})B = IB$. Thus, we have

$$\begin{aligned} B &= \mathrm{mat}(T(\mathcal{V})) \\ &= [A\mathbf{v}_1 \mid A\mathbf{v}_2 \mid A\mathbf{v}_3 \mid A\mathbf{v}_4] \\ &= \begin{bmatrix} -29 & -5 & -5 & -5 \\ 95 & 19 & 19 & 19 \\ 447 & 87 & 87 & 87 \\ -95 & -19 & -19 & -19 \\ 418 & 82 & 82 & 82 \\ 263 & 37 & 37 & 37 \end{bmatrix}, \end{aligned}$$

using a calculator for the computations.

The formula for the action of T in coordinates is

$$[T(\mathbf{s})]_{\mathcal{E}} = [T]_{\mathcal{E}\mathcal{V}}[\mathbf{s}]_{\mathcal{V}} = B[\mathbf{s}]_{\mathcal{V}}.$$

Of course, $[\mathbf{v}]_{\mathcal{E}} = \mathbf{v}$, so this equation becomes

$$T(\mathbf{s}) = B[\mathbf{s}]_{\mathcal{V}}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{T} & \mathbb{R}^6 \\ {\scriptstyle [\cdot]_{\mathcal{V}}}\big\downarrow & & \big\downarrow{\scriptstyle [\cdot]_{\mathcal{E}}=I} \\ \mathbb{R}^4 & \xrightarrow[B]{} & \mathbb{R}^6 \end{array}$$

where the vertical arrows are one-to-one correspondences. We conclude that $\mathbf{s} \in K$ if and only is $B[\mathbf{s}]_{\mathcal{V}} = 0$. In other words, $[\cdot]_{\mathcal{V}}$ gives a one-to-one correspondence between K and the nullspace of B.

A calculator computation gives the RREF of B as

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Call the variables $x_1, \dots, x_4$. We see that x_3 and x_4 are free variables, say $x_3 = \alpha$ and $x_4 = \beta$. The second row of R gives the equation $x_2 + x_3 + x_4 = 0$, so $x_2 = -\alpha - \beta$. The top row of R gives $x_1 = 0$. Thus, the nullspace of B is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

So, a basis for the nullspace of B is given by

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for K is given by the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ such that $[\mathbf{s}_1]_{\mathcal{V}} = \mathbf{w}_1$ and $[\mathbf{s}_2]_{\mathcal{V}} = \mathbf{w}_2$. We calculate that

$$\begin{aligned} \mathbf{s}_1 &= \mathcal{V}[\mathbf{s}_1]_{\mathcal{V}} \\ &= \mathcal{V}\mathbf{w}_1 \\ &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \\ &= -\mathbf{v}_2 + \mathbf{v}_3 \\ &= -\begin{bmatrix} 2 \\ 9 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} \mathbf{s}_2 &= \mathcal{V}[\mathbf{s}_2]_{\mathcal{V}} \\ &= \mathcal{V}\mathbf{w}_2 \\ &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \\ &= -\mathbf{v}_2 + \mathbf{v}_4 \\ &= \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$