

70 pts.

**Problem 1.** Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -12 & -4 & 1 & 11 \\ 1 & 1 & -1 & 0 & 0 & 5 \\ 4 & 3 & -6 & -1 & 1 & 17 \\ 2 & -1 & -8 & -3 & 1 & 3 \\ 2 & 1 & -4 & -1 & 2 & 6 \end{bmatrix}.$$

The RREF of  $A$  is the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A. Find a basis for the nullspace of  $A$ .
  - B. Find a basis for the row space of  $A$ .
  - C. Find a basis for the column space of  $A$ .
  - D. What is the rank of  $A$ ?
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50 pts.

**Problem 2.** Let  $S$  be the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 8 \\ 0 \\ 3 \end{bmatrix}.$$

- A. Cut down the list of vectors above to a basis for  $S$ . What is the dimension of  $S$ ?
- B. For each of the following vectors, determine if the vector is in  $S$  and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} 18 \\ 12 \\ 5 \\ 7 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 15 \\ 2 \\ 9 \\ 3 \end{bmatrix}$$

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50 pts.

**Problem 3.** Let  $A$  be a  $6 \times 8$  matrix and let  $B$  be a  $7 \times 7$  matrix.

- A. What is the largest possible value of the rank of  $A$ ?
  - B. If the nullspace of  $A$  has dimension 5, what is the rank of  $A$ ?
  - C. If the rowspace of  $B$  has dimension 4, what is the dimension of the nullspace of  $B$ ?
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40 pts.

**Problem 4.** Let

$$A = \begin{bmatrix} -4 & 4 \\ -3 & 4 \end{bmatrix}.$$

Find the characteristic polynomial of  $A$  and the eigenvalues of  $A$ .

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60 pts.

**Problem 5.** In each part you are given an matrix  $A$  and its eigenvalues. Find a basis for each of the eigenspaces of  $A$ . Determine if  $A$  is diagonalizable, and if it is, find a matrix  $P$  and a diagonal matrix  $D$  so that  $P^{-1}AP = D$ .

A.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \quad \text{Eigenvalues} = 1, 2.$$

B.

$$A = \begin{bmatrix} 8 & 9 & 9 \\ 0 & 2 & 0 \\ -6 & -9 & -7 \end{bmatrix}, \quad \text{Eigenvalues} = -1, 2.$$

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80 pts.

**Problem 6.**

Let  $\mathcal{U} = [\mathbf{u}_1 \ \mathbf{u}_2]$  be the ordered basis of  $\mathbb{R}^2$  where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- A. Find the change of basis matrices  $S_{\mathcal{E}\mathcal{U}}$  and  $S_{\mathcal{U}\mathcal{E}}$ .  
B. Let  $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation such that

$$\begin{aligned} L(\mathbf{u}_1) &= 2\mathbf{u}_1 - 1\mathbf{u}_2 \\ L(\mathbf{u}_2) &= 3\mathbf{u}_1 - 5\mathbf{u}_2. \end{aligned}$$

Find  $[L]_{\mathcal{U}\mathcal{U}}$ , the matrix of  $L$  with respect to the basis  $\mathcal{U}$ .

- C. Find the matrix of  $L$  with respect to the standard basis  $\mathcal{E}$  of  $\mathbb{R}^2$ .  
D. Let  $\mathbf{v}$  be the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- a. Express  $\mathbf{v}$  as a linear combination of  $\mathbf{u}_1$  and  $\mathbf{u}_2$ .  
b. Express  $T(\mathbf{v})$  as a linear combination of  $\mathbf{u}_1$  and  $\mathbf{u}_2$ .  
c. Express  $T(\mathbf{v})$  as a column vector.  
d. Check the last part against  $[L]_{\mathcal{E}\mathcal{E}}\mathbf{v}$ .
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40 pts.

**Problem 7.** Let  $S$  be the subspace of  $\mathbb{R}^6$  spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 18 \\ 38 \\ 11 \\ 11 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Let  $A$  be the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 & 1 & -3 \\ -2 & 3 & -4 & 1 & -5 & 12 \\ -8 & 15 & -22 & 3 & -25 & 56 \\ 2 & -3 & 4 & -1 & 5 & -12 \\ -8 & 14 & -20 & 3 & -24 & 53 \\ 7 & 9 & -25 & -2 & 1 & 18 \end{bmatrix}.$$

Define  $K$  by

$$K = \{\mathbf{v} \in S \mid A\mathbf{v} = \mathbf{0}\}.$$

Find a basis of  $K$ . Explain your reasoning.

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# EXAM

Exam # 2  
Take-home Exam

Math 3351, Spring 2003

Feb. 28, 2003

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- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g.,  $\sqrt{2}$ , not 1.414).
- This exam has 7 problems. There are **390 points total**.

Good luck!