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**EXAM**

Exam # 1

Math 3351, Spring 2003

Feb. 19, 2003

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ANSWERS



70 pts.

**Problem 1.** Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & 3 & 1 & -17 \\ 1 & 0 & -2 & 3 & 0 & -2 \\ 2 & 2 & -2 & 4 & 1 & -12 \\ 1 & 2 & 0 & 1 & 1 & -10 \\ 2 & 1 & -3 & 5 & 1 & -7 \end{bmatrix}$$

The RREF of  $A$  is the matrix

$$R = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -2 \\ 0 & 1 & 1 & -1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Find a basis for the nullspace of  $A$ .

*Answer:*

Call the variables  $x_1, \dots, x_6$ . From  $R$  we see that  $x_1, x_2$  and  $x_5$  are leading variables and  $x_3, x_4$  and  $x_6$  are free variables, say

$$x_3 = \alpha$$

$$x_4 = \beta$$

$$x_6 = \gamma.$$

The third row of  $R$  gives us the equation  $x_5 + 2x_6 = 0$ , so  $x_5 = -2x_6 = -2\gamma$ . The second row of  $R$  gives the equation  $x_2 + x_3 - x_4 - 5x_6 = 0$ , so  $x_2 = -x_3 + x_4 + 5x_6 = -\alpha + \beta + 5\gamma$ . Finally, the first row of  $R$  gives the equation  $x_1 - 2x_3 + 3x_4 - 2x_6 = 0$ , so  $x_1 = 2x_3 - 3x_4 + 2x_6 = 2\alpha - 3\beta + 2\gamma$ . Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2\alpha - 3\beta + 2\gamma \\ -\alpha + \beta + 5\gamma \\ \alpha \\ \beta \\ -2\gamma \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

So, the three vectors

$$\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 5 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

form a basis for the nullspace of  $A$ .

B. Find a basis for the row space of  $A$ .

*Answer:*

The nonzero rows in  $R$  form a basis for the row space of  $R$ , which is the same as the row space of  $A$ . Thus, a basis of the row space of  $A$  is given by

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -2 \end{bmatrix}, \\ \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & -5 \end{bmatrix}, \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

C. Find a basis for the column space of  $A$

*Answer:*

The leading entries in  $R$  are in columns 1, 2 and 5. For a basis of the column space of  $A$ , we take the corresponding columns of  $A$ . Thus, the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

form a basis of the column space of  $A$ .

D. What is the rank of  $A$ ?

*Answer:*

The rank of  $A$  is the common value of the dimension of the row space and the dimension of the column space. In this case, these spaces have three basis vectors, and so have dimension three. Thus, the rank of  $A$  is three.

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50 pts.

**Problem 2.** Let  $S$  be the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

A. Pare down the list of vectors above to a basis for  $S$ . What is the dimension of  $S$ ?

- B. For each of the following vectors, determine if the vector is in  $S$  and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} 12 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

*Answer:*

We can solve the problem in one step by putting all the vectors in a matrix, say

$$A = [ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \mid \mathbf{w}_1 \mid \mathbf{w}_2 ] = \left[ \begin{array}{cccc|cc} 2 & 4 & 0 & 4 & 12 & 10 \\ 1 & 0 & -2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 & 4 & 2 \\ 1 & 1 & -1 & 2 & 7 & 5 \end{array} \right],$$

The RREF of  $A$  is

$$R = [ \mathbf{r}_1 \mid \mathbf{r}_2 \mid \mathbf{r}_3 \mid \mathbf{r}_4 \mid \mathbf{r}_5 \mid \mathbf{r}_6 ] = \left[ \begin{array}{cccc|cc} 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

From this, we see that  $\mathbf{r}_1$ ,  $\mathbf{r}_2$  and  $\mathbf{r}_4$  are linearly independent and  $\mathbf{r}_3$  is a linear combination of  $\mathbf{r}_1$ ,  $\mathbf{r}_2$  and  $\mathbf{r}_4$ . The same must be true of the columns of  $A$ , so we conclude that  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_4$  are a basis of  $S$ . Since there are three vectors in a basis for  $S$ , the dimension of  $S$  is 3.

Looking at  $R$ , we see that  $\mathbf{r}_5 = 2\mathbf{r}_1 - \mathbf{r}_2 + 3\mathbf{r}_4$ . The same must be true in  $A$ , so we conclude that  $\mathbf{w}_1 = 3\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_4$  and so  $\mathbf{w}_1 \in S$ .

Looking at  $R$ , we see that  $\mathbf{r}_6$  is not a linear combination of  $\mathbf{r}_1, \dots, \mathbf{r}_4$  (because of the last row). The same must be true in  $A$ , so  $\mathbf{w}_2$  is not a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_4$ . Thus,  $\mathbf{w}_2 \notin S$ .

50 pts.

**Problem 3.** Let  $A$  be a  $9 \times 7$  matrix and let  $B$  be a  $6 \times 9$  matrix.

- A. What is the largest possible value of the rank of  $A$ ?

*Answer:*

The rank is the common value of the dimension of the row space and the dimension of the column space. In this case, there are seven columns, so the dimension of the column space must be less than or equal to 7, thus the maximum possible rank is 7.

B. If the nullspace of  $A$  has dimension 3, what is the rank of  $A$ ?

*Answer:*

The rank theorem says

$$\text{rank}(A) + \dim(N(A)) = \text{number of columns in } A.$$

The number of columns is 7 and the dimension of the nullspace is 3, so the rank of  $A$  is 4.

C. If the column space of  $B$  has dimension 4, what is the dimension of the nullspace of  $B$ ?

*Answer:*

Since the column space has dimension 4, the rank is 4. Since there are 9 columns in  $B$ , the rank theorem says

$$\text{rank}(B) + \dim(N(B)) = 9,$$

so we conclude that  $\dim(N(B)) = 5$ .

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40 pts.

**Problem 4.** Let

$$A = \begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix}.$$

Find the characteristic polynomial of  $A$  and the eigenvalues of  $A$ .

*Answer:*

The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 16 - \lambda & -21 \\ 10 & -13 - \lambda \end{bmatrix}\right) \\ &= (16 - \lambda)(-13 - \lambda) - (-21)(10) \\ &= -208 - 16\lambda + 13\lambda + 210 \\ &= \lambda^2 - 3\lambda + 2. \end{aligned}$$

The eigenvalues of  $A$  are the roots of  $p(\lambda)$ . We have  $p(\lambda) = (\lambda - 2)(\lambda - 1)$ , so the eigenvalues of  $A$  are 1 and 2.

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60 pts.

**Problem 5.** In each part you are given an matrix  $A$  and its eigenvalues. Find a basis for each of the eigenspaces of  $A$ . Determine if  $A$  is diagonalizable, and if it is, find a matrix  $P$  and a diagonal matrix  $D$  so that  $P^{-1}AP = D$ .

A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 3 \\ -2 & -2 & 4 \end{bmatrix}, \quad \text{Eigenvalues} = 1, 2.$$

*Answer:*

First consider the eigenvalue  $\lambda = 1$ . In this case we have

$$A - \lambda I = A - I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & 3 \\ -2 & -2 & 3 \end{bmatrix}.$$

Using a calculator, the RREF of this matrix is

$$\begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Call the variables  $x$ ,  $y$  and  $z$ . We see from the RREF that  $y$  and  $z$  are free variables, say  $y = \alpha$  and  $z = \beta$ . The top row of the last matrix gives us the equation  $x + y - (3/2)z = 0$ , so  $x = -y + (3/2)z = -\alpha + (3/2)\beta$ . Thus, the nullspace of  $A - I$  is parametrised by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\alpha + \frac{3}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for the  $\lambda = 1$  eigenspace of  $A$  is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}.$$

To avoid fractions, we can multiply the second vector by 2 to get the alternative basis

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

for the  $\lambda = 1$  eigenspace of  $A$ .

Next, consider the eigenvalue  $\lambda = 2$ . In this case we have

$$A - \lambda I = A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 3 \\ -2 & -2 & 2 \end{bmatrix}.$$

The calculator gives the RREF of  $A - 2I$  to be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,  $z$  is a free variable, say  $z = \alpha$ . The second row of the RREF gives us the equation  $y - z = 0$ , so  $y = z = \alpha$ . The top row of the RREF gives the equation  $x = 0$ . Thus, the nullspace of  $A - 2I$  is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the  $\lambda = 2$  eigenspace of  $A$  is one dimensional with basis vector

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We've now found three linearly independent eigenvectors of the  $3 \times 3$  matrix  $A$ , so we conclude that  $A$  is **diagonalizable**. To get the diagonalizing matrix  $P$ , put the eigenvectors we've found in as columns of  $P$ , so we can take  $P$  to be

$$P = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

We will then have  $P^{-1}AP = D$ , where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

(each column contains the eigenvalue for the corresponding column of  $P$ .)

B.

$$A = \begin{bmatrix} -12 & 3 & 7 \\ -14 & 4 & 8 \\ -13 & 3 & 8 \end{bmatrix}, \quad \text{Eigenvalues} = -2, 1.$$

*Answer:*

For the  $\lambda = 1$  eigenspace, we want to find the nullspace of

$$A - \lambda I = A - I = \begin{bmatrix} -13 & 3 & 7 \\ -14 & 3 & 8 \\ -13 & 3 & 7 \end{bmatrix}.$$

By calculator, the RREF of  $A - I$  is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that  $z$  is a free variable, say  $z = \alpha$ . The second row gives the equation  $y - 2z = 0$ , so  $y = 2\alpha$ . The top row gives the equation  $x - z = 0$ , so  $x = z = \alpha$ . Thus, the nullspace of  $A - I$  is parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus, the  $\lambda = 1$  eigenspace of  $A$  is one dimensional with basis vector

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Next, consider the  $\lambda = 2$  eigenspace. In this case, we have

$$A - \lambda I = A + 2I = \begin{bmatrix} -10 & 3 & 7 \\ -14 & 6 & 8 \\ -13 & 3 & 10 \end{bmatrix}.$$

The calculator gives

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

for the RREF of  $A + 2I$ . We see that  $z$  is free, say  $z = \alpha$ . The second row gives the equation  $y - z = 0$ , so  $y = z = \alpha$ . The top row gives the equation  $x - z = 0$ , so  $x = z = \alpha$ . Thus, the nullspace of  $A + 2I$  is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the  $\lambda = -2$  eigenspace is one dimensional with basis vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since we've found bases for all the eigenspaces but have not produced 3 linearly independent eigenvectors, we conclude that  $A$  is **not diagonalizable**.

60 pts.

**Problem 6.**

Let  $\mathcal{U} = [\mathbf{u}_1 \quad \mathbf{u}_2]$  be the ordered basis of  $\mathbb{R}^2$  where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

A. Find the change of basis matrices  $S_{\mathcal{E}\mathcal{U}}$  and  $S_{\mathcal{U}\mathcal{E}}$ .

*Answer:*

The change of basis matrix  $S_{\mathcal{E}\mathcal{U}}$  is defined by  $\mathcal{U} = \mathcal{E}S_{\mathcal{E}\mathcal{U}}$ . In matrix terms this means  $\text{mat}(\mathcal{U}) = IS_{\mathcal{E}\mathcal{U}}$ , so

$$S_{\mathcal{E}\mathcal{U}} = \text{mat}(\mathcal{U}) = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

We then have

$$S_{\mathcal{U}\mathcal{E}} = (S_{\mathcal{E}\mathcal{U}})^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix},$$

using a calculator to find the inverse.

B. Let  $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation such that

$$L(\mathbf{u}_1) = 2\mathbf{u}_1 - 4\mathbf{u}_2$$

$$L(\mathbf{u}_2) = 3\mathbf{u}_1.$$

Find  $[L]_{\mathcal{U}\mathcal{U}}$ , the matrix of  $L$  with respect to the basis  $\mathcal{U}$ .

*Answer:*

The defining equation for  $[L]_{\mathcal{U}\mathcal{U}}$  is  $L(\mathcal{U}) = \mathcal{U}[L]_{\mathcal{U}\mathcal{U}}$ . We have

$$L(\mathcal{U}) = [L(\mathbf{u}_1) \quad L(\mathbf{u}_2)] = [2\mathbf{u}_1 - 4\mathbf{u}_2 \quad 3\mathbf{u}_1] = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix},$$

so we conclude that

$$[L]_{\mathcal{U}\mathcal{U}} = \begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix}.$$

C. Find the matrix of  $L$  with respect to the standard basis  $\mathcal{E}$  of  $\mathbb{R}^2$

*Answer:*

The change of basis formula for the matrix of a linear transformation says, in this case,

$$[L]_{\mathcal{E}\mathcal{E}} = S_{\mathcal{E}\mathcal{U}}[L]_{\mathcal{U}\mathcal{U}}S_{\mathcal{U}\mathcal{E}}.$$

We've found all these matrices above, so we have

$$[L]_{\mathcal{E}\mathcal{E}} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -22 & 36 \\ -15 & 24 \end{bmatrix},$$

using a calculator for the computation.