
EXAM

Exam #1

Math 1351-04, Spring 2003

February 17, 2002

ANSWERS

90 pts.

Problem 1. Figure 1 shows the graph of a function f . In each part, find the right-hand limit, left-hand limit and two-sided limit as x approaches a ; find the value $f(a)$; and determine if the function is continuous at a . Justify your answers.

A. $a = -2$.

Answer:

We have

$$\lim_{x \rightarrow -2^-} f(x) = 5$$

$$\lim_{x \rightarrow -2^+} f(x) = 5$$

and so

$$\lim_{x \rightarrow -2} f(x) = 5.$$

From the graph, $f(-2) = 1$. The function is not continuous at -2 , since

$$\lim_{x \rightarrow -2} f(x) \neq f(-2).$$

B. $a = 0$.

Answer:

We have

$$\lim_{x \rightarrow 0^-} f(x) = 3$$

$$\lim_{x \rightarrow 0^+} f(x) = 3$$

and so

$$\lim_{x \rightarrow 0} f(x) = 3.$$

We also have $f(0) = 3$, so the function is continuous at $x = 0$.

C. $a = 2$.

Answer:

We have

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2^+} f(x) = 4$$

and so

$$\lim_{x \rightarrow 2} f(x), \quad \text{does not exist.}$$

The value of $f(2)$ is $f(2) = 2$. The function is not continuous at $x = 2$, because $\lim_{x \rightarrow 2} f(x)$ does not exist.

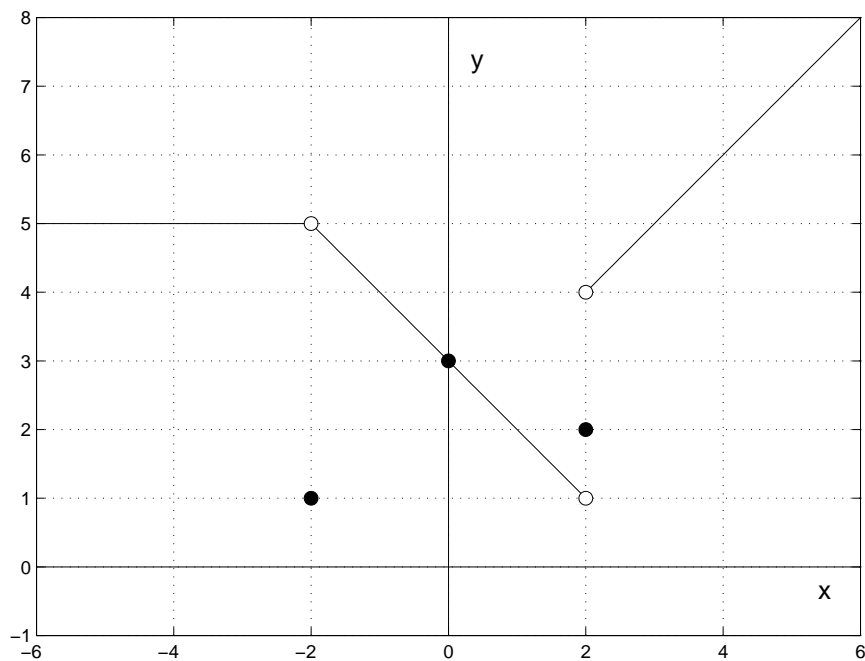


Figure 1: The function for Problem 1

50 pts.

Problem 2. Consider the function

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ 2, & x = 0 \\ x + 1, & 0 < x \leq 2 \\ x^2 - 1, & x > 2. \end{cases}$$

Find the suspicious points and determine if f is continuous at each suspicious point. Explain your answers.

Answer:

The suspicious points are $x = 0$ and $x = 2$, because the formulas change at these points. At $x = 2$ we have

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (x^2 + 1) = 0^2 + 1 = 1 \\ \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (x + 1) = 0 + 1 = 1 \end{aligned}$$

and hence

$$\lim_{x \rightarrow 0} f(x) = 1.$$

However, $f(0) = 2$, so the function is *not* continuous at $x = 0$, because

$$\lim_{x \rightarrow 0} f(x) \neq f(0).$$

For the suspicious point $x = 2$, we have

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x + 1) = 2 + 1 = 3 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 - 1) = (2)^2 - 1 = 3\end{aligned}$$

and so

$$\lim_{x \rightarrow 2} f(x) = 3.$$

We also have $f(2) = 2 + 1 = 3$, so the function is continuous at $x = 2$, since

$$\lim_{x \rightarrow 2} f(x) = f(2).$$

80 pts.

Problem 3. In each part, find the limit (if it exists).

A.

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1}$$

Answer:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 2)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 2) \\ &= 1 + 2 = 3.\end{aligned}$$

B.

$$\lim_{x \rightarrow 0} \frac{x}{\sin(5x)}$$

Answer:

We calculate as follows:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x}{\sin(5x)} &= \lim_{x \rightarrow 0} \frac{5x}{5 \sin(5x)} \\ &= \frac{1}{5} \lim_{x \rightarrow 0} \frac{5x}{\sin(5x)} \\ &= \frac{1}{5} \lim_{x \rightarrow 0} \frac{1}{\frac{\sin(5x)}{5x}} \\ &= \frac{1}{5} \frac{1}{1} \\ &= \frac{1}{5},\end{aligned}$$

using the limit

$$\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1 \quad (*)$$

from class.

C.

$$\lim_{x \rightarrow 0} \frac{\tan(x)}{\tan(2x)}$$

Answer:

Putting in lots of detail, we can calculate as follows:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan(x)}{\tan(2x)} &= \lim_{x \rightarrow 0} \frac{\frac{\sin(x)}{\cos(x)}}{\frac{\sin(2x)}{\cos(2x)}} \\ &= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(x)}{\sin(2x) \cos(x)} \\ &= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(x)}{\cos(x) \sin(2x)} \\ &= \lim_{x \rightarrow 0} \frac{\cos(2x) \sin(x) \frac{1}{x}}{\cos(x) \sin(2x) \frac{1}{x}} \\ &= \lim_{x \rightarrow 0} \frac{\cos(2x) \frac{\sin(x)}{x}}{\cos(x) \frac{\sin(2x)}{x}} \\ &= \lim_{x \rightarrow 0} \frac{\cos(2x) \frac{\sin(x)}{x}}{\cos(x) \frac{x}{2 \frac{\sin(x)}{2x}}}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1} \frac{1}{2(1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Using the limit (*) and the fact that \cos is a continuous function with $\cos(0) = 1$.

D.

$$\lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - 3}{x-2}$$

Answer:

We make use of the identity $(a-b)(a+b) = a^2 - b^2$ and calculate as follows:

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - 3}{x-2} &= \lim_{x \rightarrow 2} \frac{\sqrt{2x+5} - 3}{x-2} \frac{\sqrt{2x+5} + 3}{\sqrt{2x+5} + 3} \\
&= \lim_{x \rightarrow 2} \frac{(\sqrt{2x+5})^2 - 3^2}{(x-2)(\sqrt{2x+5} + 3)} \\
&= \lim_{x \rightarrow 2} \frac{2x + 5 - 9}{(x-2)(\sqrt{2x+5} + 3)} \\
&= \lim_{x \rightarrow 2} \frac{2x - 4}{(x-2)(\sqrt{2x+5} + 3)} \\
&= \lim_{x \rightarrow 2} \frac{2(x-2)}{(x-2)(\sqrt{2x+5} + 3)} \\
&= \lim_{x \rightarrow 2} \frac{2}{\sqrt{2x+5} + 3} \\
&= \frac{2}{\sqrt{2(2) + 5} + 3} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

40 pts.

Problem 4. Solve the following equation:

$$\ln(x) + \ln(3x+2) = 0.$$

Answer:

We solve the equation by the following sequence of steps:

$$\begin{aligned}\ln(x) + \ln(3x + 2) &= 0 && (*) \\ \ln(x(3x + 2)) &= 0 && \text{product rule for logs} \\ e^{\ln(x(3x+2))} &= e^0 && \text{take exponential of both sides} \\ x(3x + 2) &= 1 && \text{inverse relations, } e^0 = 1 \\ 3x^2 + 2x &= 1 \\ 3x^2 + 2x - 1 &= 0 \\ (3x - 1)(x + 1) &= 0\end{aligned}$$

The solutions of the last equation are $x = -1$ and $x = 1/3$. However, if we plug $x = -1$ into the original equation (*), the first term becomes $\ln(-1)$, which is undefined. Thus, $x = -1$ is *not* a solution of the original equation (*). If we plug $x = 1/3$ into (*), both of the expressions inside the logs are positive, and so defined. We conclude that the only solution of (*) is $x = 1/3$.

40 pts.

Problem 5. Evaluate each of the following exactly.

1. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Answer:

If we let $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, then we know that

$$\begin{aligned}\cos(\theta) &= -\sqrt{2}/2 \\ 0 &\leq \theta \leq \pi.\end{aligned}$$

We know $\cos(\pi/4) = \sqrt{2}/2$, so the angle $\pi/4$ gives us a point

$$(\cos(\pi/4), \sin(\pi/4)) = (\sqrt{2}/2, \sqrt{2}/2)$$

on the unit circle with x -coordinate $\sqrt{2}/2$. See Figure 2. Reflecting this point through the y -axis gives us a point with x -coordinate $-\sqrt{2}/2$. Hence the angle θ we want is the one marked in the diagram, which is $\theta = \pi - \pi/4 = 3\pi/4$.

2. $\sin(\tan^{-1}(x))$.

Answer:

Let $\theta = \tan^{-1}(x)$, so $\tan(\theta) = x$. Label the angle θ in the reference triangle and label the sides to reflect $\tan(\theta) = x$. See Figure 3. We can find the

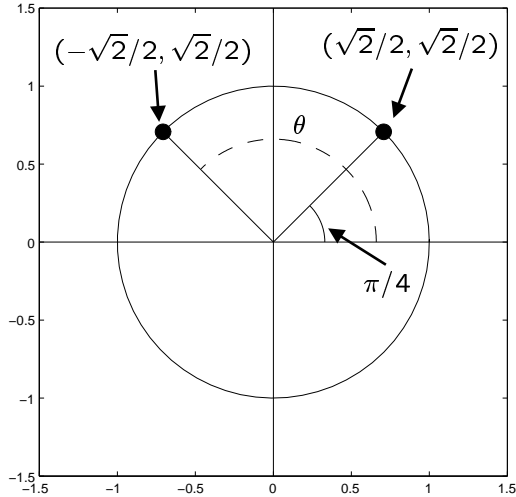


Figure 2: Figure for Problem 5, part 1

length c of the hypotenuse by the Pythagorean theorem: $c^2 = 1^2 + x^2$, so $c = \sqrt{1 + x^2}$. Since $\sin(\theta)$ is the opposite side over the hypotenuse, we have

$$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1 + x^2}}.$$

40 pts.

Problem 6. A bank pays 5% interest per year, compounded continuously. Suppose that you open a savings account at this bank with a deposit of \$100, and make no further deposits.

1. How much money will be in the account after 3 years? (Give your answer rounded to the nearest penny.)

Answer:

The formula for continuously compounded interest is

$$A = Pe^{rt}.$$

In this case the interest rate is $5\% = 5/100 = 0.05$ and the principal is $P = 100$, so the formula is

$$A = 100e^{0.05t}.$$

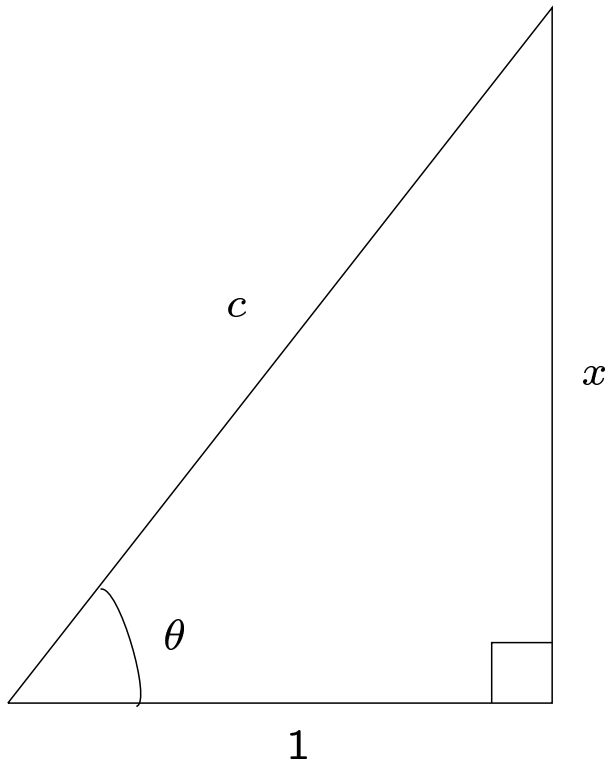


Figure 3: The Figure for Problem 5, part 2

Plugging in $t = 3$ and using a calculator gives

$$A = 100e^{0.05(3)} \approx \$116.18$$

to the nearest penny.

2. How long will it take until there is \$300 in the account? (Give your answer in years, rounded to 2 decimal places.)

Answer:

From above, the formula is

$$A = 100e^{0.05t}.$$

To determine when we have \$300 in the account, set $A = 300$ and solve for t . Thus, we have to solve

$$300 = 100e^{0.05t}.$$

We proceed as follows

$$300 = 100e^{0.05t}$$

$$3 = e^{0.05t}$$

$$\ln(3) = \ln(e^{0.05t}) = 0.05t$$

$$t = \frac{\ln(3)}{0.05} \approx 21.97$$

years, rounded to two decimal places.
