
EXAM

Practice for Third Exam

Math 1352-006, Fall 2003

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ANSWERS

Problem 1. In each part determine if the series is convergent or divergent. If it is convergent find the sum. (These are geometric or telescoping series.)

A.

$$\sum_{k=3}^{\infty} \frac{5}{2^k}.$$

Answer:

We have

$$\sum_{k=3}^{\infty} \frac{5}{2^k} = \frac{5}{2^3} + \frac{5}{2^4} + \frac{5}{2^5} + \cdots$$

This is a geometric series with ratio $r = 1/2$. Since $|r| < 1$, the series is convergent. The first term is $a = 5/2^3$, so the sum of the series is

$$\frac{a}{1-r} = \frac{5/8}{1-1/2} = 2\left(\frac{5}{8}\right) = \frac{5}{4}$$

B.

$$\sum_{k=1}^{\infty} 3e^k.$$

Answer:

We have

$$\sum_{k=1}^{\infty} 3e^k = 3e + 3e^2 + 3e^3 + \cdots$$

This is a geometric series with ratio $r = e$. Since $|r| = e > 1$, the series is divergent.

C.

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}2}{7^{2k}}.$$

Answer:

We have

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}2}{7^{2k}} = \frac{2}{7^2} - \frac{2}{7^4} + \frac{2}{7^6} - \frac{2}{7^8} + \cdots$$

This is a geometric series with ratio $r = -1/7^2 = -1/49$. Since $|r| = 1/49 < 1$, the series is convergent. The first term is $a = 2/7^2$, so the sum is

$$\frac{a}{1-r} = \frac{2/49}{1-(-1/49)} = \frac{2/49}{1+1/49} = \frac{2/49}{50/49} = \frac{49}{50}\frac{2}{49} = \frac{2}{50} = 1/25$$

D.

$$\sum_{k=3}^{\infty} \frac{1}{k(k+1)}.$$

Answer:

Partial fractions gives

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

(check). Thus, we can write

$$\sum_{k=3}^{\infty} \frac{1}{k(k+1)} = \sum_{k=3}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right].$$

Noting that the sum starts at $k = 3$, we have the following for the partial sums.

$$\begin{aligned}
S_1 &= \frac{1}{3} - \frac{1}{4} \\
S_2 &= \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = \frac{1}{3} - \frac{1}{5} \\
S_3 &= \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} = \frac{1}{3} - \frac{1}{6} \\
&\vdots \\
S_n &= \frac{1}{3} - \frac{1}{n+3}
\end{aligned}$$

Since

$$\lim_{n\to\infty} S_n = \frac{1}{3} - 0 = 1/3,$$

we conclude that the series converges and that the sum is 1/3.

Problem 2. In each part, express the given number as a single fraction.

A. $0.77\overline{7}$

Answer:

We have

$$\begin{aligned}
0.77\overline{7} &= 0.777777\cdots \\
&= 0.7 + 0.07 + 0.007 + 0.0007 + \cdots \\
&= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \cdots
\end{aligned}$$

This is a geometric series with ratio $r = 1/10$ (so it's convergent) and first term $a = 7/10$. Thus, we have

$$\begin{aligned}
0.77\overline{7} &= \frac{a}{1-r} \\
&= \frac{7/10}{1-1/10} \\
&= \frac{\frac{7}{10}}{\frac{9}{10}} \\
&= \frac{10}{9}\frac{7}{10} \\
&= \frac{7}{9}.
\end{aligned}$$

B. $23.542121\overline{21}$

Answer:

We can write

$$23.542121\overline{21} = 23.54 + 0.002121\overline{21}. \tag{1}$$

For the repeating decimal, we have

$$\begin{aligned}
0.002121\overline{21} &= 0.0021 + 0.000021 + 0.00000021 + \cdots \\
&= \frac{21}{10^4} + \frac{21}{10^6} + \frac{21}{10^8} + \cdots
\end{aligned}$$

This is a geometric series with ratio $r = 1/10^2 = 1/100$ ($|r| < 1$, so it's convergent) and first term $21/10^4$. Thus, we have

$$\begin{aligned}
0.002121\overline{21} &= \frac{a}{1-r} \\
&= \frac{\frac{21}{10^4}}{1-\frac{1}{100}} \\
&= \frac{\frac{21}{10^4}}{\frac{99}{100}} \\
&= \frac{100}{99}\frac{21}{10^4} \\
&= \frac{21}{99(100)} \\
&= \frac{7}{33(100)} \\
&= \frac{7}{3300}.
\end{aligned}$$

Of course, we also have

$$23.54 = \frac{2354}{100},$$

and so, by (1), we have

$$23.54212\overline{121} = \frac{2354}{100} + \frac{7}{3300} = \frac{77689}{3300}.$$

Problem 3.

A rubber ball is dropped from a height of 12 feet. After each bounce, the ball goes up to 3/4 of the distance it previously fell. What is the total vertical distance traveled by the ball?

Answer:
The ball falls 12 feet to the floor. On the first bounce it goes up to a height of (3/4)(12). Thus, on this bounce, the ball travels (3/4)(12) up and travels (3/4)(12) down, for a total distance of 2(3/4)(12). On the second bounce it goes up to a height of $(3/4)[(3/4)(12)] = (3/4)^2(12)$. Thus, it travels $(3/4)^2(12)$ up and $(3/4)^2(12)$ down for a total distance of $2(3/4)^2(12)$ on the second bounce. The distance traveled on the third bounce is $2(3/4)^3(12)$ and so forth.

Thus, the total distance traveled is

$$D = 12 + 2(3/4)(12) + 2(3/4)^2(12) + 2(3/4)^3(12) + \cdots \tag{2}$$

The series

$$2(3/4)(12)+2(3/4)^2(12)+2(3/4)^3(12)+\cdots = 24(3/4)+24(3/4)^2+24(3/4)^3+\cdots$$

is a geometric series with $r = 3/4$ (and so it's convergent) and first term $a = 24(3/4) = 18$. Thus, the sum of the series is

$$\begin{aligned}\frac{a}{1-r} &= \frac{18}{1-3/4}\\ &= \frac{18}{1/4}\\ &= 4(18)\\ &= 72.\end{aligned}$$

Thus, from (2), we have

$$D = 12 + 72 = 84.$$

Problem 4. In each part, determine if the series is convergent or divergent. Be sure to say which test you are using and to show the details of how the test is applied.

Part 1.

$$\sum_{k=1}^{\infty} e^{-3k}.$$

Answer:

We have

$$\sum_{k=1}^{\infty} e^{-3k} = e^{-3} + e^{-6} + e^{-9} + \cdots$$

This is a geometric series with $r = e^{-3}$. Since $|r| < 1$, the series is convergent. The sum is

$$\frac{a}{1-r} = \frac{e^{-3}}{1-e^{-3}}.$$

Part 2.

$$\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}.$$

Answer:

This is a p-series with $p = 2/3$. By the p-Series Test (p. 519) the series is divergent, since $p = 2/3 \leq 1$.

Part 3.

$$\sum_{k=1}^{\infty} \frac{k^2}{k^8 + k^5 + 7}. \tag{3}$$

Answer:

Keeping just the fastest growing terms on the top and bottom, we have

$$\frac{k^2}{k^8 + k^5 + 7} \sim \frac{k^2}{k^8} = \frac{1}{k^6}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^6} \tag{4}$$

is a p-series with $p = 6 > 1$, so is convergent. We use the Limit Comparison Test (p. 523) to compare the series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{k^8 + k^5 + 7} \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^6}.$$

We have

$$\begin{aligned}
\frac{a_k}{b_k} &= \frac{\dfrac{k^2}{k^8+k^5+7}}{\dfrac{1}{k^6}} \\
&= \frac{k^8}{k^8+k^5+7} \\
&= \frac{1}{1+1/k^3+7/k^8} \\
&\to 1 = L.
\end{aligned}$$

Since $0 < L < \infty$, the Limit Comparison Test says that both series converge or both diverge. Since we know that the series (4) is convergent, we conclude that the series (3) is convergent.

Part 4.

$$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}.$$

Answer:

We apply the Integral Test (p. 516) with $f(x) = 1/(x\ln(x))$. For $x \geq 2$, this function is positive. It is clear enough that f is decreasing on $[2, \infty)$ since both x and $\ln(x)$ are increasing functions. Thus, we can apply the integral test. We need to check the integral

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \int_2^{\infty} \frac{1}{\ln(x)}\frac{1}{x}\,dx.$$

Make the substitution $u = \ln(x)$. Then we have $du = dx/x$, so we have

$$\int_2^{\infty} \frac{1}{\ln(x)}\frac{1}{x}\,dx = \int_{\ln(2)}^{\infty} \frac{1}{u}\,du.$$

where we have changed the limits is accordance with $u = \ln(x)$. We have

$$\begin{aligned}
\int_{\ln(2)}^{\infty} \frac{1}{u}\,du &= \ln(u)\Big|_{u=\ln(2)}^{u=\infty} \\
&= [\lim_{u\to\infty} \ln(u)] - \ln(\ln(2)) \\
&= \infty.
\end{aligned}$$

So this integral diverges. The Integral Test tells us that the series and the integral and the series both converge or both diverge. We conclude that the series is divergent.

Part 5.

$$\sum_{k=1}^{\infty} \frac{k^2}{2k^2+1}.$$

Answer:

We have

$$\lim_{k\to\infty} \frac{k^2}{2k^2+1} = \lim_{k\to\infty} \frac{1}{2+1/k^2} = \frac{1}{2}$$

Since the k-th term of the series does not approach 0 as $k \to \infty$, the Divergence Test (p. 515) says that our series is divergent.

Part 6.

$$\sum_{k=1}^{\infty} \frac{k^3}{2^k}.$$

Answer:

Apply the Ratio Test (p. 527). The k-th term a_k is given by

$$a_k = \frac{k^3}{2^k}$$

and so we have

$$a_{k+1} = \frac{(k+1)^3}{2^{k+1}}$$

Then we have

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\frac{(k+1)^3}{2^{k+1}}}{\frac{k^3}{2^k}} \\
&= \frac{2^k}{k^3}\frac{(k+1)^3}{2^{k+1}} \\
&= \frac{1}{2}\left(\frac{k+1}{k}\right)^3 \\
&= \frac{1}{2}(1+1/k)^3 \\
&\to \frac{1}{2}(1+0)^3 \\
&= \frac{1}{2} = r.
\end{aligned}$$

Since $r < 1$, the Ratio Test tells us the series is convergent.

Part 7.

$$\sum_{k=1}^{\infty} \left(\frac{k}{2k+1} \right)^k.$$

Answer:

Apply the Root Test (p. 530). In the notation of the Root Test, we have

$$a_k = \left(\frac{k}{2k+1} \right)^k.$$

Then we have

$$\begin{aligned} \sqrt[k]{a_k} &= \left[\left(\frac{k}{2k+1} \right)^k \right]^{1/k} \\ &= \frac{k}{2k+1} \\ &= \frac{1}{2+1/k} \\ &\to \frac{1}{2+0} \\ &= 1/2 = r. \end{aligned}$$

Since $r = 1/2 < r$, the Root Test tells us the series is convergent.

Part 8.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k + k}}. \tag{5}$$

Answer:

Keeping only the fastest growing term on the bottom, have

$$\frac{1}{\sqrt{2^k + k}} \sim \frac{1}{\sqrt{2^k}} = \frac{1}{[\sqrt{2}]^k}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{[\sqrt{2}]^k} \tag{6}$$

is a geometric series with ratio $r = 1/\sqrt{2}$. Since $|r| < 1$, we conclude that (6) is convergent.

Now, we apply the Limit Comparison Test with

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k + k}}. \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{[\sqrt{2}]^k}.$$

We have

$$\begin{aligned}
\frac{a_k}{b_k} &= \frac{\frac{1}{\sqrt{2^k+k}}}{\frac{1}{\sqrt{2^k}}} \\
&= \frac{\sqrt{2^k}}{\sqrt{2^k+k}} \\
&= \sqrt{\frac{2^k}{2^k+k}} \\
&= \sqrt{\frac{1}{1+k/2^k}} \\
&\to \sqrt{\frac{1}{1+0}} \\
&= 1 = L,
\end{aligned}$$

where we've used $k/2^k \to 0$ as $k \to \infty$. Since $0 < L < \infty$, the Limit Comparison Test tells us that either both series converge or both diverge. Since we know (6) is convergent, we conclude that (5) is convergent.

Part 9.

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}.$$

Answer:

Use the Ratio Test. We have

$$a_k = \frac{2^k}{k!}$$

and so

$$a_{k+1} = \frac{2^{k+1}}{(k+1)!}$$

Thus, we can compute

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\dfrac{2^{k+1}}{(k+1)!}}{\dfrac{2^k}{k!}} \\
&= \frac{k!}{2^k} \frac{2^{k+1}}{(k+1)!} \\
&= 2 \frac{k!}{(k+1)!} \\
&= 2 \frac{1 \cdot 2 \cdot 3 \cdots k}{1 \cdot 2 \cdot 3 \cdots k \cdot (k+1)} \\
&= \frac{2}{k+1} \\
&\to 0 = r.
\end{aligned}$$

Since $r = 0 < 1$, the Ratio Test tells us the series is convergent.

Part 10.

$$\sum_{k=1}^{\infty} e^{-5/k}.$$

Answer:

Since

$$\lim_{k\to\infty} \frac{5}{k} = 0,$$

we have

$$\lim_{k\to\infty} e^{-5/k} = e^0 = 1.$$

Since the k-th term of the series does not go to zero as $k \to \infty$, the Divergence Test tells us the series is divergent.

Part 11.

$$\sum_{k=1}^{\infty} \frac{3^{2k+1}}{(2k+1)!}.$$

Answer:

Apply the Ratio Test. We have

$$a_k = \frac{3^{2k+1}}{(2k+1)!}$$

and so

$$a_{k+1} = \frac{3^{2(k+1)+1}}{(2(k+1)+1)!} = \frac{3^{2k+3}}{(2k+3)!}.$$

Thus, we can compute

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\dfrac{3^{2k+3}}{(2k+3)!}}{\dfrac{3^{2k+1}}{(2k+1)!}} \\
&= \frac{(2k+1)!}{3^{2k+1}} \frac{3^{2k+3}}{(2k+3)!} \\
&= 3^2 \frac{(2k+1)!}{(2k+3)!} \\
&= 9 \frac{1 \cdot 2 \cdot 3 \cdots (2k+1)}{1 \cdot 2 \cdot 3 \cdots (2k+1)(2k+2)(2k+3)} \\
&= \frac{9}{(2k+2)(2k+3)} \\
&\to 0 = r.
\end{aligned}$$

Since $r = 0 < 1$, the Ratio Test tells us the series is convergent.

Part 12.

$$\sum_{k=2}^{\infty} \frac{1}{[\ln(k)]^k}.$$

Answer:

Use the Root Test. We compute

$$\begin{aligned}
\sqrt[k]{a_k} &= \left[\frac{1}{[\ln(k)]^k}\right]^{1/k} \\
&= \frac{1}{\ln(k)} \\
&\to 0 = r.
\end{aligned}$$

Since $r = 0 < 1$, the Root Test tells us the series is convergent.

Part 13.

$$\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^5 + k^3 + 2}}. \tag{7}$$

Answer:

Keeping only the fastest growing terms on top and bottom, we have

$$\frac{k^2}{\sqrt{k^5 + k^3 + 2}} \sim \frac{k^2}{\sqrt{k^5}} = \frac{k^2}{k^{5/2}} = \frac{1}{k^{5/2-4/2}} = \frac{1}{k^{1/2}}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \tag{8}$$

is a p-series with $p = 1/2 \leq 1$, and so is divergent.

We apply the Limit Comparison Test with

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^5 + k^3 + 2}} \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}.$$

We compute

$$\begin{aligned}
\frac{a_k}{b_k} &= \frac{\frac{k^2}{\sqrt{k^5+k^3+2}}}{\frac{1}{\sqrt{k}}} \\
&= \sqrt{k}\frac{k^2}{\sqrt{k^5+k^3+2}} \\
&= \sqrt{k}\frac{\sqrt{k^4}}{\sqrt{k^5+k^3+2}} \\
&= \sqrt{\frac{k^5}{k^5+k^3+2}} \\
&= \sqrt{\frac{1}{1+1/k^2+2/k^5}} \\
&\to \sqrt{\frac{1}{1+0+0}} \\
&= 1 = L.
\end{aligned}$$

Since $0 < L < \infty$, the Limit Comparison Test tells us both series converge or both series diverge. Since we know that (8) diverges, we conclude that (7) diverges.

Part 14.

$$\sum_{k=1}^{\infty} \frac{2^k}{k^{10}}.$$

Answer:

Use the Ratio Test. We have

$$a_k = \frac{2^k}{k^{10}}$$

and so

$$a_{k+1} = \frac{2^{k+1}}{(k+1)^{10}}.$$

We compute

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\frac{2^{k+1}}{(k+1)^{10}}}{\frac{2^k}{k^{10}}} \\
&= \frac{k^{10}}{2^k} \frac{2^{k+1}}{(k+1)^{10}} \\
&= 2\left(\frac{k}{k+1}\right)^{10} \\
&= 2\left(\frac{1}{1+1/k}\right)^{10} \\
&\to 2\left(\frac{1}{1+0}\right)^{10} \\
&= 2 = r.
\end{aligned}$$

Since $r = 2 > 1$, the Ratio Test tells us the series diverges.

Part 15.

$$\sum_{k=2}^{\infty} \frac{1}{\ln(k)}. \tag{9}$$

Answer:

Use the Zero-infinity Limit Comparison Test (p. 525) with

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln(k)} \qquad \text{and} \qquad \sum_{k=2}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Of course, we know that

$$\sum_{k=2}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k} \tag{10}$$

is divergent. We compute

$$\frac{a_k}{b_k} = \frac{\frac{1}{\ln(k)}}{\frac{1}{k}}$$
$$= \frac{k}{\ln(k)}$$
$$\to \infty$$

The second part of the Zero-infinity Limit Comparison Test tells us that $\sum a_k$ is divergent if $\sum b_k$ is divergent. Since we know (10) is divergent, we conclude (9) is divergent.

Problem 5. Assume that $x > 0$. In each case, find the set of all $x > 0$ for which the series converges.

A.

$$\sum_{k=1}^{\infty} \frac{x^k}{k!}$$

Answer:

Use the ratio test, with $a_k = x^k/k!$. Then,

$$\frac{a_{k+1}}{a_k} = \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}$$
$$= \frac{k!}{x^k}\frac{x^{k+1}}{(k+1)!}$$
$$= x\frac{k!}{(k+1)!}$$
$$= x\frac{1 \cdot 2 \cdot 3 \cdots k}{1 \cdot 2 \cdot 3 \cdots k \cdot (k+1)}$$
$$= \frac{x}{k+1}$$
$$\to 0 = r.$$

Since r in the ratio test is $0 < 1$ for all x, there series converges for all $x > 0$.

B.

(11)
$$\sum_{k=1}^{\infty} \frac{x^k}{k2^k}$$

Answer:

Try the ratio test, with

$$a_k = \frac{x^k}{k2^k}.$$

Then we have

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\dfrac{x^{k+1}}{(k+1)2^{k+1}}}{\dfrac{x^k}{k2^k}} \\
&= \frac{k2^k}{x^k}\frac{x^{k+1}}{(k+1)2^{k+1}} \\
&= \frac{x^{k+1}}{x^k}\frac{2^k}{2^{k+1}}\frac{k}{k+1} \\
&= x(2)\frac{1}{1+1/k} \\
&\to x/2 = r.
\end{aligned}$$

The series converges if $r < 1$, i.e., if $x/2 < 1$ or $x < 2$. The series diverges if $x/2 = r > 1$, or $x > 2$. At $x = 2$, we have $r = 1$, so the ratio test is inconclusive. However, if we substitute $x = 2$ in the series, we get

$$\sum_{k=1}^{\infty} \frac{2^k}{k2^k} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This is the harmonic series, which we know is divergent. Thus, the series (11) is convergent for $0 < x < 2$ and divergent for $x \geq 2$.
