EXAM

Practice for Second Exam

Math 1352-006, Fall 2003

Nov 4, 2003

ANSWERS

Problem 1.

In each part, find the integral.

A.

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$$

Answer:

Make the substitution $x = 2\sin(\theta)$. Then, we have $dx/d\theta = 2\cos(\theta)$, so $dx = 2\cos(\theta)\,d\theta$. We also have

$$\begin{aligned} 4 - x^2 &= 4 - 4\sin^2(\theta) \\ &= 4[1 - \sin^2(\theta)] \\ &= 4\cos^2(\theta), \end{aligned}$$

so

$$(4-x^2)^{3/2} = 4^{3/2}[\cos^2(\theta)]^{3/2} = 8\cos^3(\theta).$$

Plugging into the integral gives us

$$\begin{aligned} \int \frac{x^2}{(4-x^2)^{3/2}}\,dx &= \int \frac{4\sin^2(\theta)}{8\cos^3(\theta)}\,2\cos(\theta)\,d\theta \\ &= \int \frac{\sin^2(\theta)}{\cos^2(\theta)}\,d\theta \\ &= \int \tan^2(\theta)\,d\theta \\ &= \tan(\theta) - \int 1\,d\theta \qquad \text{(Reduction Formula (3))} \\ (*) \qquad &= \tan(\theta) - \theta + C. \end{aligned}$$

From above, we have $x/2 = \sin(\theta)$, so $\theta = \sin^{-1}(x/2)$. We also have $\cos(\theta) = \sqrt{4-x^2}/2$, so

$$\tan(\theta) = \frac{\sin(\theta)}{\cos th} = \frac{\frac{x}{2}}{\frac{\sqrt{4-x^2}}{2}} = \frac{x}{\sqrt{4-x}}.$$

Plugging into (*), we get

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$$

B.

$$\int x\cos(2x)\,dx.$$

Answer:

Use integration by parts. We want to differentiate the x factor to get rid of it. Thus, in the integration by parts formula

$$\int uv'\,dx = uv - \int uv'\,dx \tag{1}$$

we set $u = x$ and $v' = \cos(2x)$. We then have $u' = 1$ and

$$v = \int v'\,dx = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x) \qquad \text{(why?)}$$

Plugging into (1) gives us

$$\begin{aligned}\int x\cos(2x)\,dx &= \frac{1}{2}x\sin(2x) - \int (1)\left[\frac{1}{2}\sin(2x)\right]dx \\ &= \frac{1}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx \\ &= \frac{1}{2}\sin(2x) - \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right] + C \\ &= \frac{1}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.\end{aligned}$$

C.

$$\int x^3\sqrt{1-x^2}\,dx$$

Answer:

If you write the integral as

$$\int x^2\sqrt{1-x^2}\,x\,dx,$$

you may see that the simple substitution

$$u = 1 - x^2 \tag{2}$$

will work. This substitution gives us $du/dx = -2x$ and so $x\,dx = (-1/2)\,du$. From (2) we have $x^2 = 1 - u$. Plugging into the integral gives us

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int (1-u)\sqrt{u}\left(-\frac{1}{2}\right)du \\ &= -\frac{1}{2}\int (u^{1/2} - u^{3/2})\,du \\ &= -\frac{1}{2}\left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/3}\right] + C \\ &= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C\end{aligned}$$

D.

$$\int \ln(x)\, dx$$

Answer:

Use integration by parts (Formula (1)) with $u = \ln(x)$ and so $v' = 1$. Then, $u' = 1/x$ and

$$v = \int v'\, dx = \int 1\, dx = x.$$

Plugging into the integration by parts formula(1) gives

$$\begin{aligned}\int \ln(x)\, dx &= x\ln(x) - \int \left[\frac{1}{x}\right](x)\, dx \\ &= x\ln(x) - \int 1\, dx \\ &= x\ln(x) - x + C.\end{aligned}$$

E.

$$\int x^2 \ln(x)\, dx$$

Answer:

If we use integration by parts and differentiate $\ln(x)$, all that will be left is powers of x. Thus, set $u = \ln(x)$ and $v' = x^2$. We then have $u' = 1/x$ and

$$v = \int v'\, dx = \int x^2\, dx = \frac{x^3}{3}.$$

Plugging into the integration by parts formula (1) we have

$$\begin{aligned}\int x^2 \ln(x)\, dx &= \frac{x^3}{3}\ln(x) - \int \left[\frac{1}{x}\right]\left[\frac{x^3}{3}\right] dx \\ &= \frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2\, dx \\ &= \frac{x^3}{3}\ln(x) - \frac{1}{3}\left[\frac{x^3}{3}\right] + C \\ &= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C\end{aligned}$$

F.

$$\int \sqrt{4+x^2}\,dx.$$

Answer:

Make the trigonometric substitution

$$x = 2\tan(\theta). \tag{3}$$

Then we have

$$\frac{dx}{d\theta} = 2\sec^2(\theta) \implies dx = 2\sec^2(\theta)\,d\theta.$$

We also have

$$\begin{aligned} 4+x^2 &= 4+4\tan^2(\theta) \\ &= 4[1+\tan^2(\theta)] \\ &= 4\sec^2(\theta) \end{aligned}$$

and so

$$\sqrt{4+x^2} = \sqrt{4\sec^2(\theta)} = 2\sec(\theta). \tag{4}$$

Plugging into the integral we get

$$\begin{aligned} \int \sqrt{4+x^2}\,dx &= \int 2\sec(\theta) 2\sec^2(\theta)\,d\theta \\ &= 4\int \sec^3(\theta)\,d\theta \\ &= 4\left[\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\int \sec(\theta)\,d\theta\right] \qquad \text{(Reduction Formula (5))} \\ &= 2\sec\theta\tan(\theta) + 2\ln|\sec\theta + \tan(\theta)| + C \\ &= 2\frac{\sqrt{4+x^2}}{2}\frac{x}{2} + \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C, \qquad \text{(From (3) and (4))} \\ &= \frac{1}{2}x\sqrt{4+x^2} + 2\ln\left|\frac{1}{2}\sqrt{4+x^2} + \frac{1}{2}x\right| + C \\ &= \frac{1}{2}x\sqrt{4+x^2} + 2\ln|\sqrt{4+x^2} + x| + C, \end{aligned}$$

where in the last (optional) step, we've absorbed $2\ln(1/2)$ into the arbitary constant C.

G.

$$\int \frac{\sqrt{x^2-1}}{x}\,dx$$

Answer:

Note that the integrand is defined for $x \geq 1$ and $x \leq -1$. To handle the case $x \geq 1$, make the trig substitution $x = \sec(\theta)$. We have

$$\frac{dx}{d\theta} = \sec(\theta)\tan(\theta) \implies dx = \sec(\theta)\tan(\theta)\,d\theta$$

and we have

$$x^2 - 1 = sec^2(\theta) - 1 = \tan^2(\theta)$$

and so we have $\sqrt{x^2-1} = \tan(\theta)$. Plugging into the integral gives

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\tan(\theta)}{\sec(\theta)}\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \tan^2(\theta)\,d\theta \\
&= \tan(\theta) - \theta + C, \qquad \text{(From Reduction Formula (3))} \\
&= \sqrt{x^2-1} - \sec^{-1}(x) + C
\end{aligned}$$

Thus, we have

$$(5) \qquad \int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1} - \sec^{-1}(x) + C, \qquad x \geq 1.$$

Now consider the case $x \leq -1$. To deal with this case make the substitution $x = -u$ (so $u \geq 1$) in the integral. Of course $dx = -du$ and we have

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\sqrt{(-u)^2-1}}{-u}(-1)\,du \\
&= \int \frac{\sqrt{u^2-1}}{u}\,du \\
&= \sqrt{u^2-1} - \sec^{-1}(u) + C, \qquad \text{(from (5))} \\
&= \sqrt{(-x)^2-1} - \sec^{-1}(-x) + C \\
&= \sqrt{x^2-1} - \sec^{-1}(-x) + C
\end{aligned}$$

Combining the two case, we have the formula

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \begin{cases} \sqrt{x^2-1} - \sec^{-1}(x) + C, & x \geq 1 \\ \sqrt{x^2-1} - \sec^{-1}(-x) + C, & x \leq -1. \end{cases}$$

We can easily combine the two cases into one, using absolute value. Thus, our final answer is

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1} - \sec^{-1}|x| + C.$$

H.

$$\int \frac{x-1}{x^2+2x+1}\,dx$$

Answer:

A quick check with the quadratic formula shows that x^2+2x+1 doesn't factor. Thus, we want to complete the square. That is, we want to write $x^2+2x+2=(x-h)^2+k=x^2-2xh+h^2+k$ Comparing coefficents shows that $-2h=2$ so $h=-1$ and $2=h^2+k=1+k$ so $k=1$. Thus, we have $x^2+2x+2=(x+1)^2+1$. Thus, our integral becomes

$$\int \frac{x-1}{1+(x+1)^2)}\,dx.$$

In this integral, substitute $u=x+1$. With the substitution, $dx=du$ and $x=u-1$. Thus, our integral becomes

$$\begin{aligned}
\int \frac{x-1}{1+(x+1)^2)}\,dx &= \int \frac{(u-1)-1}{1+u^2}\,du \\
&= \int \frac{u-2}{1+u^2}\,du \\
&= \int \frac{u}{1+u^2}\,du - 2\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{2}\ln(1+u^2) - 2\int \frac{du}{1+u^2}, \qquad \text{(guess and correct method)} \\
&= \frac{1}{2}\ln(1+u^2) - 2\tan^{-1}(u) + C \\
&= \frac{1}{2}\ln[1+(x+1)^2] - 2\tan^{-1}(x+1) + C \\
&= \frac{1}{2}\ln(x^2+2x+2) - 2\tan^{-1}(x+1) + C.
\end{aligned}$$

Problem 2. In each part, give the *form* of the partial fraction decomposition. This is a formula involving undetermined coefficients. **Do not find the coefficients!** (No calculation is required).

A.

$$\frac{x^3+2x+1}{(x-1)(x-2)(x+3)}$$

Answer:

$$\frac{x^3+2x+1}{(x-1)(x-2)(x+3)} = \boxed{\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+3}}.$$

B.

$$\frac{1}{x(x^2+1)}$$

Answer:

$$\frac{1}{x(x^2+1)} = \boxed{\frac{A}{x} + \frac{Bx+C}{x^2+1}}.$$

C.

$$\frac{x^4+1}{x(x^2+1)^2}$$

Answer:

$$\frac{x^4+1}{x(x^2+1)^2} = \boxed{\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}}.$$

D.

$$\frac{x^3}{(x-2)^2(x+2)^2(x-1)}$$

Answer:

$$\frac{x^3}{(x-2)^2(x+2)^2(x-1)}$$
$$= \boxed{\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{x-1}}.$$

Problem 3. In each part, find the partial fraction decomposition of the given rational function (i.e., find the coefficients).

A.

$$\frac{5x-1}{x(x-1)(x+1)}$$

Answer:

The form of the partial fraction decomposition should be

$$\frac{5x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}. \tag{6}$$

Clearing the denominators in this equation leads to the equation

$$5x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1). \tag{7}$$

Setting $x = 0$ in this equation yields the equation

$$-1 = A(-1)(1) + B(0)(1) + C(0)(-1)$$

which gives us $-1 = -A$, so $A = 1$. Setting $x = 1$ in equation (7) yields $4 = B(1)(2)$, so $B = 2$. Finally, setting $x = -1$ in equation (7) yields the equation $-6 = 2C$, so $C = -3$. Plugging these values for A, B and C into equation (6) gives the answer

$$\boxed{\frac{5x-1}{x(x-1)(x+1)} = \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1}.}$$

B.

$$\frac{x^2+1}{x^2(x-1)}$$

Answer:

The form of the partial fraction decomposition is

$$\frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}. \tag{8}$$

Clearing denominators in this equation gives

$$x^2 + 1 = Ax(x-1) + B(x-1) + Cx^2. \tag{9}$$

Expanding this out gives

$$\begin{aligned} x^2 + 1 &= Ax^2 - Ax + Bx - B + Cx^2 \\ &= (A+C)x^2 + (-A+B)x - B. \end{aligned}$$

Equating coefficents gives the system of equations

$$\begin{aligned} A + C &= 1 \\ -A + B &= 0 \\ -B &= 1 \end{aligned}$$

These are easy to solve, giving $A = -1$, $B = -1$ and $C = 2$. Plugging these values into (8) gives

$$\frac{x^2+1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}.$$

C.

$$\frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)}$$

Answer:

The form of the partial fraction decomposition is

$$\frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}. \tag{10}$$

Clearing the denominators gives

$$\begin{aligned} 2x^3 + 2x^2 - 1 &= Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2 \\ &= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 \\ &= (A + C)x^3 + (B + D)x^2 + Ax + B. \end{aligned}$$

Equating coefficents leads to the following system of equations

$$\begin{aligned} A + C &= 2 \\ B + D &= 2 \\ A &= 0 \\ B &= -1. \end{aligned}$$

These are easy to solve and the solution is $A = 0$, $B = -1$, $C = 2$ and $D = 3$. Plugging these values into equation (10) gives

$$\frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)} = -\frac{1}{x^2} + \frac{2x + 3}{x^2 + 1}.$$
