
EXAM

Practice for Second Exam

Math 1352-006, Fall 2003

Nov 4, 2003

ANSWERS

Problem 1.

In each part, find the integral.

A.

$$\int \frac{x^2}{(4-x^2)^{3/2}} dx$$

Answer:

Make the substitution $x = 2 \sin(\theta)$. Then, we have $dx/d\theta = 2 \cos(\theta)$, so $dx = 2 \cos(\theta) d\theta$. We also have

$$\begin{aligned} 4 - x^2 &= 4 - 4 \sin^2(\theta) \\ &= 4[1 - \sin^2(\theta)] \\ &= 4 \cos^2(\theta), \end{aligned}$$

so

$$(4 - x^2)^{3/2} = 4^{3/2} [\cos^2(\theta)]^{3/2} = 8 \cos^3(\theta).$$

Plugging into the integral gives us

$$\begin{aligned} \int \frac{x^2}{(4-x^2)^{3/2}} dx &= \int \frac{4 \sin^2(\theta)}{8 \cos^3(\theta)} 2 \cos(\theta) d\theta \\ &= \int \frac{\sin^2(\theta)}{\cos^2(\theta)} d\theta \\ &= \int \tan^2(\theta) d\theta \\ &= \tan(\theta) - \int 1 d\theta \quad (\text{Reduction Formula (3)}) \\ (*) \quad &= \tan(\theta) - \theta + C. \end{aligned}$$

From above, we have $x/2 = \sin(\theta)$, so $\theta = \sin^{-1}(x/2)$. We also have $\cos(\theta) = \sqrt{4-x^2}/2$, so

$$\tan(\theta) = \frac{\sin(\theta)}{\cos \theta} = \frac{\frac{x}{2}}{\frac{\sqrt{4-x^2}}{2}} = \frac{x}{\sqrt{4-x^2}}.$$

Plugging into (*), we get

$$\int \frac{x^2}{(4-x^2)^{3/2}} dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$$

B.

$$\int x \cos(2x) dx.$$

Answer:

Use integration by parts. We want to differentiate the x factor to get rid of it. Thus, in the integration by parts formula

$$(1) \quad \int uv' dx = uv - \int uv' dx$$

we set $u = x$ and $v' = \cos(2x)$. We then have $u' = 1$ and

$$v = \int v' dx = \int \cos(2x) dx = \frac{1}{2} \sin(2x) \quad (\text{why?})$$

Plugging into (1) gives us

$$\begin{aligned} \int x \cos(2x) dx &= \frac{1}{2} x \sin(2x) - \int (1) \left[\frac{1}{2} \sin(2x) \right] dx \\ &= \frac{1}{2} \sin(2x) - \frac{1}{2} \int \sin(2x) dx \\ &= \frac{1}{2} \sin(2x) - \frac{1}{2} \left[-\frac{1}{2} \cos(2x) \right] + C \\ &= \frac{1}{2} \sin(2x) + \frac{1}{4} \cos(2x) + C. \end{aligned}$$

C.

$$\int x^3 \sqrt{1-x^2} dx$$

Answer:

If you write the integral as

$$\int x^2 \sqrt{1-x^2} x dx,$$

you may see that the simple substitution

$$(2) \quad u = 1 - x^2$$

will work. This substitution gives us $du/dx = -2x$ and so $x dx = (-1/2) du$.

From (2) we have $x^2 = 1 - u$. Plugging into the integral gives us

$$\begin{aligned} \int x^3 \sqrt{1-x^2} dx &= \int (1-u) \sqrt{u} \left(-\frac{1}{2} \right) du \\ &= -\frac{1}{2} \int (u^{1/2} - u^{3/2}) du \\ &= -\frac{1}{2} \left[\frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right] + C \\ &= -\frac{1}{3} (1-x^2)^{3/2} + \frac{1}{5} (1-x^2)^{5/2} + C \end{aligned}$$

D.

$$\int \ln(x) dx$$

Answer:

Use integration by parts (Formula (1)) with $u = \ln(x)$ and so $v' = 1$. Then, $u' = 1/x$ and

$$v = \int v' dx = \int 1 dx = x.$$

Plugging into the integration by parts formula(1) gives

$$\begin{aligned} \int \ln(x) dx &= x \ln(x) - \int \left[\frac{1}{x} \right] (x) dx \\ &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + C. \end{aligned}$$

E.

$$\int x^2 \ln(x) dx$$

Answer:

If we use integration by parts and differentiate $\ln(x)$, all that will be left is powers of x . Thus, set $u = \ln(x)$ and $v' = x^2$. We then have $u' = 1/x$ and

$$v = \int v' dx = \int x^2 dx = \frac{x^3}{3}.$$

Plugging into the integration by parts formula (1) we have

$$\begin{aligned} \int x^2 \ln(x) dx &= \frac{x^3}{3} \ln(x) - \int \left[\frac{1}{x} \right] \left[\frac{x^3}{3} \right] dx \\ &= \frac{x^3}{3} \ln(x) - \frac{1}{3} \int x^2 dx \\ &= \frac{x^3}{3} \ln(x) - \frac{1}{3} \left[\frac{x^3}{3} \right] + C \\ &= \frac{x^3}{3} \ln(x) - \frac{x^3}{9} + C \end{aligned}$$

F.

$$\int \sqrt{4+x^2} dx.$$

Answer:

Make the trigonometric substitution

$$(3) \quad x = 2 \tan(\theta).$$

Then we have

$$\frac{dx}{d\theta} = 2 \sec^2(\theta) \implies dx = 2 \sec^2(\theta) d\theta.$$

We also have

$$\begin{aligned} 4+x^2 &= 4+4 \tan^2(\theta) \\ &= 4[1+\tan^2(\theta)] \\ &= 4 \sec^2(\theta) \end{aligned}$$

and so

$$(4) \quad \sqrt{4+x^2} = \sqrt{4 \sec^2(\theta)} = 2 \sec(\theta).$$

Plugging into the integral we get

$$\begin{aligned} \int \sqrt{4+x^2} dx &= \int 2 \sec(\theta) 2 \sec^2(\theta) d\theta \\ &= 4 \int \sec^3(\theta) d\theta \\ &= 4 \left[\frac{1}{2} \sec(\theta) \tan(\theta) + \frac{1}{2} \int \sec(\theta) d\theta \right] \quad (\text{Reduction Formula (5)}) \\ &= 2 \sec \theta \tan(\theta) + 2 \ln|\sec \theta + \tan(\theta)| + C \\ &= 2 \frac{\sqrt{4+x^2}}{2} \frac{x}{2} + \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C, \quad (\text{From (3) and (4)}) \\ &= \frac{1}{2} x \sqrt{4+x^2} + 2 \ln \left| \frac{1}{2} \sqrt{4+x^2} + \frac{1}{2} x \right| + C \\ &= \frac{1}{2} x \sqrt{4+x^2} + 2 \ln|\sqrt{4+x^2} + x| + C, \end{aligned}$$

where in the last (optional) step, we've absorbed $2 \ln(1/2)$ into the arbitrary constant C .

G.

$$\int \frac{\sqrt{x^2-1}}{x} dx$$

Answer:

Note that the integrand is defined for $x \geq 1$ and $x \leq -1$. To handle the case $x \geq 1$, make the trig substitution $x = \sec(\theta)$. We have

$$\frac{dx}{d\theta} = \sec(\theta) \tan(\theta) \implies dx = \sec(\theta) \tan(\theta) d\theta$$

and we have

$$x^2 - 1 = \sec^2(\theta) - 1 = \tan^2(\theta)$$

and so we have $\sqrt{x^2 - 1} = \tan(\theta)$. Plugging into the integral gives

$$\begin{aligned} \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\tan(\theta)}{\sec(\theta)} \sec(\theta) \tan(\theta) d\theta \\ &= \int \tan^2(\theta) d\theta \\ &= \tan(\theta) - \theta + C, \quad (\text{From Reduction Formula (3)}) \\ &= \sqrt{x^2 - 1} - \sec^{-1}(x) + C \end{aligned}$$

Thus, we have

$$(5) \quad \int \frac{\sqrt{x^2 - 1}}{x} dx = \sqrt{x^2 - 1} - \sec^{-1}(x) + C, \quad x \geq 1.$$

Now consider the case $x \leq -1$. To deal with this case make the substitution $x = -u$ (so $u \geq 1$) in the integral. Of course $dx = -du$ and we have

$$\begin{aligned} \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\sqrt{(-u)^2 - 1}}{-u} (-1) du \\ &= \int \frac{\sqrt{u^2 - 1}}{u} du \\ &= \sqrt{u^2 - 1} - \sec^{-1}(u) + C, \quad (\text{from (5)}) \\ &= \sqrt{(-x)^2 - 1} - \sec^{-1}(-x) + C \\ &= \sqrt{x^2 - 1} - \sec^{-1}(-x) + C \end{aligned}$$

Combining the two case, we have the formula

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \begin{cases} \sqrt{x^2 - 1} - \sec^{-1}(x) + C, & x \geq 1 \\ \sqrt{x^2 - 1} - \sec^{-1}(-x) + C, & x \leq -1. \end{cases}$$

We can easily combine the two cases into one, using absolute value. Thus, our final answer is

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \sqrt{x^2 - 1} - \sec^{-1}|x| + C.$$

H.

$$\int \frac{x-1}{x^2+2x+1} dx$$

Answer:

A quick check with the quadratic formula shows that $x^2 + 2x + 1$ doesn't factor. Thus, we want to complete the square. That is, we want to write $x^2 + 2x + 2 = (x-h)^2 + k = x^2 - 2xh + h^2 + k$. Comparing coefficients shows that $-2h = 2$ so $h = -1$ and $2 = h^2 + k = 1 + k$ so $k = 1$. Thus, we have $x^2 + 2x + 2 = (x+1)^2 + 1$. Thus, our integral becomes

$$\int \frac{x-1}{1+(x+1)^2} dx.$$

In this integral, substitute $u = x + 1$. With the substitution, $dx = du$ and $x = u - 1$. Thus, our integral becomes

$$\begin{aligned} \int \frac{x-1}{1+(x+1)^2} dx &= \int \frac{(u-1)-1}{1+u^2} du \\ &= \int \frac{u-2}{1+u^2} du \\ &= \int \frac{u}{1+u^2} du - 2 \int \frac{1}{1+u^2} du \\ &= \frac{1}{2} \ln(1+u^2) - 2 \int \frac{du}{1+u^2}, \quad (\text{guess and correct method}) \\ &= \frac{1}{2} \ln(1+u^2) - 2 \tan^{-1}(u) + C \\ &= \frac{1}{2} \ln[1+(x+1)^2] - 2 \tan^{-1}(x+1) + C \\ &= \frac{1}{2} \ln(x^2+2x+2) - 2 \tan^{-1}(x+1) + C. \end{aligned}$$

Problem 2. In each part, give the *form* of the partial fraction decomposition.

This is a formula involving undetermined coefficients. **Do not find the coefficients!** (No calculation is required).

A.

$$\frac{x^3 + 2x + 1}{(x-1)(x-2)(x+3)}$$

Answer:

$$\frac{x^3 + 2x + 1}{(x-1)(x-2)(x+3)} = \boxed{\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}}.$$

B.

$$\frac{1}{x(x^2 + 1)}$$

Answer:

$$\frac{1}{x(x^2 + 1)} = \boxed{\frac{A}{x} + \frac{Bx + C}{x^2 + 1}}.$$

C.

$$\frac{x^4 + 1}{x(x^2 + 1)^2}$$

Answer:

$$\frac{x^4 + 1}{x(x^2 + 1)^2} = \boxed{\frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}}.$$

D.

$$\frac{x^3}{(x - 2)^2(x + 2)^2(x - 1)}$$

Answer:

$$\begin{aligned} & \frac{x^3}{(x - 2)^2(x + 2)^2(x - 1)} \\ &= \boxed{\frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x + 2} + \frac{D}{(x + 2)^2} + \frac{E}{x - 1}}. \end{aligned}$$

Problem 3. In each part, find the partial fraction decomposition of the given rational function (i.e., find the coefficients).

A.

$$\frac{5x - 1}{x(x - 1)(x + 1)}$$

Answer:

The form of the partial fraction decomposition should be

$$(6) \quad \frac{5x - 1}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}.$$

Clearing the denominators in this equation leads to the equation

$$(7) \quad 5x - 1 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1).$$

Setting $x = 0$ in this equation yields the equation

$$-1 = A(-1)(1) + B(0)(1) + C(0)(-1)$$

which gives us $-1 = -A$, so $A = 1$. Setting $x = 1$ in equation (7) yields $4 = B(1)(2)$, so $B = 2$. Finally, setting $x = -1$ in equation (7) yields the equation $-6 = 2C$, so $C = -3$. Plugging these values for A , B and C into equation (6) gives the answer

$$\frac{5x - 1}{x(x - 1)(x + 1)} = \frac{1}{x} + \frac{2}{x - 1} + \frac{3}{x + 1}.$$

B.

$$\frac{x^2 + 1}{x^2(x - 1)}$$

Answer:

The form of the partial fraction decomposition is

$$(8) \quad \frac{x^2 + 1}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}.$$

Clearing denominators in this equation gives

$$(9) \quad x^2 + 1 = Ax(x - 1) + B(x - 1) + Cx^2.$$

Expanding this out gives

$$\begin{aligned} x^2 + 1 &= Ax^2 - Ax + Bx - B + Cx^2 \\ &= (A + C)x^2 + (-A + B)x - B. \end{aligned}$$

Equating coefficients gives the system of equations

$$\begin{aligned} A + C &= 1 \\ -A + B &= 0 \\ -B &= 1 \end{aligned}$$

These are easy to solve, giving $A = -1$, $B = -1$ and $C = 2$. Plugging these values into (8) gives

$$\frac{x^2 + 1}{x^2(x - 1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x - 1}.$$

C.

$$\frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)}$$

Answer:

The form of the partial fraction decomposition is

$$(10) \quad \frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}.$$

Clearing the denominators gives

$$\begin{aligned} 2x^3 + 2x^2 - 1 &= Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2 \\ &= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 \\ &= (A + C)x^3 + (B + D)x^2 + Ax + B. \end{aligned}$$

Equating coefficients leads to the following system of equations

$$\begin{aligned} A + C &= 2 \\ B + D &= 2 \\ A &= 0 \\ B &= -1. \end{aligned}$$

These are easy to solve and the solution is $A = 0$, $B = -1$, $C = 2$ and $D = 3$. Plugging these values into equation (10) gives

$$\frac{2x^3 + 2x^2 - 1}{x^2(x^2 + 1)} = -\frac{1}{x^2} + \frac{2x + 3}{x^2 + 1}.$$
