EXAM

Exam #2

Math 2360, Second Summer Session, 2002

August 2, 2002

ANSWERS

70 pts.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 4 & 9 \\ 0 & -3 & -3 & -1 & -5 & -5 \\ -1 & 2 & 3 & -1 & 1 & 8 \\ -1 & 4 & 5 & 2 & 9 & 9 \\ 1 & -10 & -11 & -12 & -35 & -11 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Find a basis for the nullspace of A.

Answer:

Call the variables $x_1, \dots x_6$. Note that A and R have the same nullspace. Looking at R we see that the variables x_1, x_2 and x_4 are leading variables and x_3, x_5 and x_6 are free variables, say $x_3 = \alpha$, $x_5 = \beta$ and $x_6 = \gamma$. Reading the non-zero rows of R from the bottom up, we have the following equations

$$\begin{aligned} x_4 + 2x_5 - x_6 = 0 &\implies x_4 = -2\beta + \gamma \\ x_2 + x_3 + x_5 + 2x_6 = 0 &\implies x_2 = -\alpha - \beta - 2\gamma \\ x_1 - x_3 - x_5 - 3x_6 = 0 &\implies x_1 = \alpha + \beta + 3\gamma. \end{aligned}$$

Thus, the nullspace of A is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \alpha + \beta + 3\gamma \\ -\alpha - \beta - 2\gamma \\ \alpha \\ -2\beta + \gamma \\ \beta \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, a basis of the nullspace of A is given by the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

B. Find a basis for the rowspace of A.

Answer:

A basis of the rowspace of A is given by the nonzero rows in the RREF of A, i.e., R. Thus, a basis of the rowspace of A is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}.$$

C. Find a basis for the columnspace of A

Answer:

The columns in R that contain the leading entries are 1, 2 and 4. These form a basis for the columnspace of R. Since the columns of R and A satisfy the same linear relations, the corresponding columns of A form a basis of the columnspace of A. Thus, columns 1, 2 and 4 of A form a basis of the columnspace of A. Explicitly, a basis for the columnspace of A is given by

$$\begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \\ 2 \\ 4 \\ -10 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \\ -12 \end{bmatrix}.$$

D. What is the rank of A?

Answer:

The rank is the dimension of the rowspace and the columnspace of A (which must have the same dimension). We've found that these spaces have 3 basis vectors, so they have dimension 3. Thus, $\operatorname{rank}(A) = 3$.

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50 pts.

Problem 2. Let A be a 6×8 matrix and let B be a 7×7 matrix.

A. What is the largest possible value of the rank of A?

Answer: 6.

B. If the nullspace of A has dimension 5, what is the rank of A?

Answer:

The rank theorem says the rank plus the nullity of A must equal the number of columns. The number of columns is 8, and we're given that the nullity is 5, so we must have $\operatorname{rank}(A) = 3$

C. If the rowspace of B has dimension 4, what is the dimension of the nullspace of B?

Answer:

The rank is equal to the dimension of the rowspace or columnspace. Thus, we know $\text{rank}(B) = 4$. The rank plus the nullity of B is equal to 7 (the number of columns). Thus, the nullity of B is 3. By definition, the nullity is the dimension of the nullspace, so the nullspace of B has dimension 3.

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50 pts.

Problem 3. In each part, determine if the given vectors in $\mathbb{R}^5$ are linearly independent. Justify your answer. If the vectors are linearly dependent, find scalars c_1, c_2, c_3 and c_4, not all zero, so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

A.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Answer:

Let's put the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ is a matrix A. Thus,

$$A = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \right] = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 4 \\ 2 & 1 & 0 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix}.$$

The Reduced Row Echelon Form of A is

$$R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c|c|c} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 \end{array} \right],$$

where we have labeled the columns of R. Looking at the columns of R, we see that

$$\mathbf{r}_4 = -2\mathbf{r}_1 + 3\mathbf{r}_2 - \mathbf{r}_3$$

or, to put it another way,

$$2\mathbf{r}_1 - 3\mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 = \mathbf{0}.$$

Since the columns of A satisfy the same linear relations as the columns of R, we must have

$$2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

Thus, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are **linearly dependent**.

B.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

Answer:

Again, put the vectors into a matrix A. Thus,

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 5 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

The Reduced Row Echelon Form of A is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the columns of R are linearly independent. Since the columns of A satisfy the same linear relations as the columns of R, we conclude that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are **linearly independent**.

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50 pts.

Problem 4. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^4$. In each part, determine if the given vector is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and, if so, express it as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

A.

$$\mathbf{w}_1 = \begin{bmatrix} 9 \\ 6 \\ 4 \\ 3 \end{bmatrix}.$$

B.

$$\mathbf{w}_2 = \begin{bmatrix} 7 \\ 7 \\ 6 \\ 6 \end{bmatrix}$$

Answer:
To solve both parts at once, put the vectors in a matrix, say

$$A = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{w}_1 \mid \mathbf{w}_2\ ] = \left[\begin{array}{ccc|cc} 4 & 3 & 2 & 9 & 7 \\ 3 & 2 & 2 & 6 & 7 \\ 2 & 2 & 2 & 4 & 6 \\ 2 & 1 & 2 & 3 & 6 \end{array}\right],$$

where we've put in the bar to separate the $\mathbf{v}$'s from the $\mathbf{w}$'s. The Reduced Row Echelon Form of A is

$$R = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = [\ \mathbf{r}_1 \mid \mathbf{r}_2 \mid \mathbf{r}_3 \mid \mathbf{r}_4 \mid \mathbf{r}_5\ ].$$

From R, we can read off the relation $\mathbf{r}_4 = 2\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3$. The same relation must hold between the columns of A, so we have $\mathbf{w}_1 = 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, so $\mathbf{w}_1$ is in $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Also from R, we wee that $\mathbf{r}_5$ is not a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The same must be true for the columns of A. Thus, $\mathbf{w}_2$ is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In other words, $\mathbf{w}_2 \notin \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

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40 pts.

Problem 5. In each case, determine if the given vectors span $\mathbb{R}^3$ and, if so, pare the given list of vectors down to a basis of $\mathbb{R}^3$.

A.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Answer:

Put the vectors in a matrix

$$A = [\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \mid \mathbf{v}_5\ ] = \begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

The RREF of A is

$$R = \left[\ \mathbf{r}_1 \mid \mathbf{r}_2 \mid \mathbf{r}_3 \mid \mathbf{r}_4 \mid \mathbf{r}_5\ \right] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We see that columns $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_4$ are linearly independent, so the same must be true for $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$. Since three independent vectors in $\mathbb{R}^3$ must form a basis, we conclude that the given set of vectors spans $\mathbb{R}^3$ and that one way to pare it down to a basis is $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$.

B.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}.$$

Answer:

Put the vectors in a matrix

$$A = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \mid \mathbf{v}_5\ \right] = \begin{bmatrix} 1 & 2 & 1 & 0 & 2 \\ 4 & 8 & 3 & -1 & 7 \\ 2 & 4 & 1 & -1 & 3 \end{bmatrix}.$$

The RREF of A is

$$R = \left[\ \mathbf{r}_1 \mid \mathbf{r}_2 \mid \mathbf{r}_3 \mid \mathbf{r}_4 \mid \mathbf{r}_5\ \right] = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that $\mathbf{r}_1$ and $\mathbf{r}_3$ are linearly independent and the rest of the columns of R are linear combinations of $\mathbf{r}_1$ and $\mathbf{r}_3$. The same must be true of A, so $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly independent and all of the other $\mathbf{v}_j$'s are combinations of these two. Thus, $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_5)$ has dimension two, and so can not be equal to all of $\mathbb{R}^3$. Thus, the given vectors do not span $\mathbb{R}^3$.

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50 pts. **Problem 6**. Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

A. Pare down the list of vectors above to a basis for S. What is the dimension of S?

B. For each of the following vectors, determine if the vector is in S and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} 12 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

Answer:
We can solve the problem in one step by putting all the vectors in a matrix, say

$$A = \left[\ \mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \mathbf{v}_4 \mid \mathbf{w}_1 \mid \mathbf{w}_2\ \right] = \left[\begin{array}{cccc|cc} 2 & 4 & 0 & 4 & 12 & 10 \\ 1 & 0 & -2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 & 4 & 2 \\ 1 & 1 & -1 & 2 & 7 & 5 \end{array}\right],$$

The RREF of A is

$$R = \left[\ \mathbf{r}_1 \mid \mathbf{r}_2 \mid \mathbf{r}_3 \mid \mathbf{r}_4 \mid \mathbf{r}_5 \mid \mathbf{r}_6\ \right] = \left[\begin{array}{cccc|cc} 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

From this, we see that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_4$ are linearly independent and $\mathbf{r}_3$ is a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_4$. The same must be true of the columns of A, so we conclude that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ are a basis of S. Since there are three vectors in a basis for S, the dimension of S is 3.

Looking at R, we see that $\mathbf{r}_5 = 2\mathbf{r}_1 - \mathbf{r}_2 + 3\mathbf{r}_4$. The same must be true in A, so we conclude that $\mathbf{w}_1 = 3\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_4$ and so $\mathbf{w}_1 \in S$.

Looking at R, we see that $\mathbf{r}_6$ is not a linear combination of $\mathbf{r}_1, \dots, \mathbf{r}_4$ (because of the last row). The same must be true in A, so $\mathbf{w}_2$ is not a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_4$. Thus, $\mathbf{w}_2 \notin S$.

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50 pts. **Problem 7**. In this problem we work in the vector space $P_3 = \{\, p(x) = ax^2 + bx + c \mid a, b, c \in \mathbb{R}\,\}$, the space of polynomials of degree < 3. In each part, determine is the given subset of P_3 is a subspace of P_3. You must justify your answers.

A. The set of polynomials $p(x) \in P_3$ such that $p(0) = 0$.

Answer:

This is a subspace of P_3. To verify this, we check the three properties necessary for a subspace. Let $S = \{\, p(x) \in P_3 \mid p(0) = 0\,\}$.

1.) S contains 0.

Proof. In fact, the zero polynomial is zero at every point x, including $x = 0$. □

2.) S is closed under scalar multiplication.

Proof. Suppose that $p(x) \in S$, so $p(0) = 0$. If α is a scalar, then $(\alpha p)(0) = \alpha \cdot p(0) = \alpha \cdot 0 = 0$, so $\alpha p(x) \in S$. □

3.) S is closed under addition.

Proof. Suppose that $p(x), q(x) \in S$, so $p(0) = 0$ and $q(0) = 0$. Let $r(x) = p(x) + q(x)$. Then $r(0) = p(0) + q(0) = 0 + 0 = 0$, so $r(x) \in S$. □

B. The set of polynomials $p(x) \in P_3$ such that $p(5) = 1$.

Answer:

This is not a subspace, because it does not contain the zero function.

C. The set of polynomials $p(x) \in P_3$ such that $p(0)p(1) = 0$.

Answer:

Let $S = \{\, p(x) \in P_3 \mid p(0)p(1) = 0 \,\}$. This is not a subspace, because it is not closed under addition. To see this, suppose that $p(x), q(x) \in S$, so

$$p(0)p(1) = 0, \quad q(0)q(1) = 0. \tag{$*$}$$

Now let $r(x) = p(x) + q(x)$. Does $r(0)r(1)$ necessarily equal 0? To check this, we compute as follows

$$\begin{aligned} r(0)r(1) &= (p(0) + q(0))(p(1) + q(1)) \\ &= p(0)p(1) + p(0)q(1) + q(0)p(1) + q(0)p(0) \\ &= p(0)q(1) + q(0)p(1) \qquad \text{using } (*), \end{aligned}$$

but this need not be zero. For example we could have $p(0) = 0$, $p(1) = 10$, $q(0) = 15$ and $q(1) = 0$. Thus, we may have $p(x) + q(x) \notin S$, so S is not a subspace.

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40 pts.

Problem 8. In this problem, we will work in the vector space

$$P_3 = \{\, ax^2 + bx + c \mid a, b, c \in \mathbb{R} \,\},$$

the space of polynomials of degree less than 3. Let $\mathcal{P}$ be the ordered basis $\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$ of P_3.

Let $T\colon P_3 \to P_3$ be the linear transformation defined by

$$T(p(x)) = p(x) - 2p'(x).$$

Find $[T]_{\mathcal{P}\mathcal{P}}$, the matrix of T with respect to the ordered basis $\mathcal{P}$.

Answer:
The defining equation of $[T]_{\mathcal{P}\mathcal{P}}$ is

$$T(\mathcal{P}) = \mathcal{P}[T]_{\mathcal{P}\mathcal{P}}$$

where

$$T(\mathcal{P}) = \begin{bmatrix} T(x^2) & T(x) & T(1) \end{bmatrix}.$$

Using the definition of T, we have

$$\begin{gathered} T(x^2) = x^2 - 2(x^2)' = x^2 - 2(2x) = x^2 - 4x \\ T(x) = x - 2(x)' = x - 2(1) = x - 2 \\ T(1) = 1 - 2(1)' = 1 - 2(0) = 1. \end{gathered}$$

Thus, we need to find the matrix A such that

$$\begin{bmatrix} x^2 - 4x & x - 2 & 1 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} A.$$

We can easily read off that

$$\begin{bmatrix} x^2 - 4x & x - 2 & 1 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Thus, we have

$$[T]_{\mathcal{P}\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

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40 pts. **Problem 9**. In this problem, we will work in the vectorspace

$$P_3 = \{\, ax^2 + bx + c \mid a, b, c \in \mathbb{R} \,\},$$

the space of polynomials of degree less than 3. Let $\mathcal{P}$ be the ordered basis $\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$ of P_3 and let $\mathcal{Q}$ be the ordered basis

$$\mathcal{Q} = \begin{bmatrix} x^2 + 2x + 2 & 2x^2 + 5x + 2 & x^2 + 2x + 1 \end{bmatrix}$$

of P_3.

A. Find the change of basis matrices $S_{\mathcal{PQ}}$ and $S_{\mathcal{QP}}$.

Answer:

The change of basis matrix $S_{\mathcal{PQ}}$ is defined by the equation

$$\mathcal{Q} = \mathcal{P} S_{\mathcal{PQ}}.$$

We have

$$\mathcal{Q} = \begin{bmatrix} x^2+2x+2 & 2x^2+5x+2 & x^2+2x+1 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

so, we must have

$$S_{\mathcal{PQ}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

We can find $S_{\mathcal{QP}}$ because $S_{\mathcal{QP}} = S_{\mathcal{PQ}}^{-1}$. Using a calculator to find the inverse, we get

$$S_{\mathcal{QP}} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ 6 & -2 & -1 \end{bmatrix}.$$

B. Let $p(x) = x^2+2$. Find $[p(x)]_{\mathcal{Q}}$, the coordinate vector of $p(x)$ with respect to the ordered basis $\mathcal{Q}$. Show how $p(x)$ can be written as a linear combination to the elements of $\mathcal{Q}$

Answer:

First we find $[p(x)]_{\mathcal{P}}$. The defining equation for this is

$$p(x) = \mathcal{P}[p(x)]_{\mathcal{P}}.$$

Since

$$p(x) = x^2 + 2 = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

, we conclude

$$[p(x)]_{\mathcal{P}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

To get $[p(x)]_{\mathcal{Q}}$, we use the change of coordinates equation

$$[p(x)]_{\mathcal{Q}} = S_{\mathcal{QP}}[p(x)]_{\mathcal{P}},$$

so we get

$$[p(x)]_{\mathcal{Q}} = S_{\mathcal{QP}}[\mathcal{PQ}]_{=} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ 6 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

To express $p(x)$ as a linear combination of the basis $\mathcal{Q}$, we have

$$\begin{aligned} p(x) &= \mathcal{Q}[p(x)]_{\mathcal{Q}} \\ &= \begin{bmatrix} x^2+2x+2 & 2x^2+5x+2 & x^2+2x+1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \\ &= (x^2+2x+2) - 2(x^2+5x+2) + 4(x^2+2x+1). \end{aligned}$$

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40 pts.

Problem 10. Let $\mathcal{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$ be the standard ordered basis of $\mathbb{R}^2$ and let $\mathcal{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$ be the ordered basis of $\mathbb{R}^2$ given by

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{EU}}$ and $S_{\mathcal{UE}}$.

Answer:

The defining equation of $S_{\mathcal{EU}}$ is

$$\mathcal{U} = \mathcal{E} S_{\mathcal{EU}}.$$

This is equivalent to the matrix equation

$$\mathrm{mat}(\mathcal{U}) = \mathrm{mat}(\mathcal{E}) S_{\mathcal{EU}} = I S_{\mathcal{EU}} = S_{\mathcal{EU}}.$$

Thus we have

$$S_{\mathcal{EU}} = \mathrm{mat}(\mathcal{U}) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

To find $S_{\mathcal{UE}}$, we use the relation $S_{\mathcal{UE}} = S_{\mathcal{EU}}^{-1}$. Using a calculator to compute the inverse, we have

$$S_{\mathcal{UE}} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

B. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such $[T]_{\mathcal{E}\mathcal{E}}$, the matrix of T with respect to the standard basis of $\mathbb{R}^2$, is given by

$$\begin{bmatrix} -7 & 9 \\ -6 & 8 \end{bmatrix}.$$

Find $[T]_{\mathcal{U}\mathcal{U}}$, the matrix of T with respect the ordered basis $\mathcal{U}$.

Answer:

We use the change of basis equation for linear transformations, which in this case is

$$\begin{aligned} [T]_{\mathcal{U}\mathcal{U}} &= S_{\mathcal{U}\mathcal{E}}[T]_{\mathcal{E}\mathcal{E}}S_{\mathcal{E}\mathcal{U}} \\ &= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -7 & 9 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$
