on $BC(X)$, the space of bounded continuous functions on the topological space X,

Problem Set

Final Exam

Math 5323, Spring 2001

May 3, 2002

ANSWERS

Problem 1. Suppose that $f \in L^1(\mathbb{R})$ and that

$$\int_J f\,dm = 0$$

for all intervals $J \subseteq \mathbb{R}$. Show that $f = 0$ a.e. Remark: f is in general complex valued.

Answer:
By considering the real and imaginary parts of f, it will suffice to consider the case where f is real valued. Since f is not assumed to be nonnegative, there is some work to do. For a fixed interval J we have

$$0 = \int_J f\,dm = \int_J f^+\,dm - \int_J f^-\,dm$$

This just says that

$$\int_J f^+\,dm = \int_J f^-\,dm$$

not that either of these integrals is zero. We need to somehow use the fact that this hold for *all* intervals.

There are a large number of ways to do this problem. Perhaps the most direct way is to use a differentiation theorem.

For example, the Lebesgue Differentiation Theorem (Theorem 3.21, p. 98) says that if $f \in L^1_{\text{loc}}$ then for every x in the Lebesgue set of f—in particular, for almost every x—we have

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|\,dy = 0$$

and

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,dy = f(x)$$

for every family $\{\, E_r \,\}_{r>0}$ that shrinks nicely to x. In particular, we can take $E_r = \mathrm{B}(r, x)$.

If we apply this theorem in the present case, we have for almost all x,

$$\begin{aligned} f(x) &= \lim_{r\to 0} \frac{1}{m(\mathrm{B}(r,x))} \int_{\mathrm{B}(r,x)} f(y)\,dy \\ &= \lim_{r\to 0} \frac{1}{2r} 0 \\ &= 0, \end{aligned}$$

since $\mathrm{B}(r, x) = (x - r, x + r)$ is an interval. Thus, $f = 0$ a.e.

A very similar idea is to use part of the Fundamental Theorem of Calculus for Lebesgue integrals, specifically Corollary 3.33 on p. 105. This corollary says that if $f \in L^1(m)$ and we define

$$F(x) = \int_{-\infty}^{x} f(t)\,dt,$$

then F is NBV and is absolutely continuous and $F' = f$ a.e. In the current case

$$\int_{-\infty}^{x} f(t)\,dt = \int_{(-\infty,x)} f\,dm = 0,$$

so $F \equiv 0$. Then $F' \equiv 0$ and $f = F'$ a.e., so $f = 0$ a.e.

Problem 2. Let X be a topological space. If $A, B \subseteq X$, show that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Answer:
Recall that if $E \subseteq X$, $\overline{E}$ is the smallest closed set that contains E, i.e., $E \subseteq \overline{E}$, $\overline{E}$ is closed, and if C is a closed set such that $E \subseteq C$, then $\overline{E} \subseteq C$.

We can apply this in the present situation as follows. First note that $A \subseteq A \cup B \subseteq \overline{A \cup B}$, and $\overline{A \cup B}$ is closed. Thus, by the first paragraph, $\overline{A} \subseteq \overline{A \cup B}$. The same argument shows that $\overline{B} \subseteq \overline{A \cup B}$, so we can conclude that

$$(*) \qquad \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$$

To get the reverse inclusion, note that $A \subseteq \overline{A} \subseteq \overline{A} \cup \overline{B}$, so $A \subseteq \overline{A} \cup \overline{B}$. Similarly, $B \subseteq \overline{A} \cup \overline{B}$. Thus, $A \cup B \subseteq \overline{A} \cup \overline{B}$. But $\overline{A}$ and $\overline{B}$ are closed and a finite union of closed sets is closed. Thus, $\overline{A} \cup \overline{B}$ is closed, and we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$$

by the first paragraph. Combining this with $(*)$, we get

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Problem 3. Recall that $C([0,1])$, equipped with the uniform norm, is a Banach space.

Suppose that $k \in C([0,1] \times [0,1])$. If $f \in C([0,1])$, define a function Tf by

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\,dy$$

Show that $Tf \in C([0,1])$ and that $T \colon C([0,1]) \to C([0,1])$ is a bounded linear operator.

Let

$$B = \{\, f \in C([0,1]) \mid \|f\|_u \le 1 \,\}$$

be the closed unit ball in $C([0,1])$. Show that $T(B)$ is pre-compact. Hint: Arzelà-Ascoli

Answer:
Recall that the uniform norm on $BC(X)$, the space of bounded continuous functions $f\colon X \to \mathbb{C}$ (where X is a topological space), is defined by

$$\|f\|_u = \sup \{\, |f(x)| \mid x \in X \,\}.$$

In the case of a compact space like $[0,1]$ or $[0,1] \times [0,1]$, the space of bounded continuous functions is the same as the space of all continuous functions.

It's clear that the integral that defines $(Tf)(x)$ exists, since the integrand is a continuous function on $[0,1]$.

First, we prove that Tf is continuous. Let $M = \|k\|_u$. Fix $p \in [0,1]$. We want to show that Tf is continuous at p. Let $\{\, x_n \,\}_{n=1}^{\infty} \subseteq [0,1]$ be any sequence that converges to p. We want to show that $(Tf)(x_n) \to (Tf)(p)$. By definition, we have

$$(Tf)(x_n) = \int_0^1 k(x_n, y) f(y)\, dy.$$

For each fixed y we have $k(x_n, y)f(y) \to k(p, y)f(y)$ by the continuity of k. We also have

$$|k(x_n, y)f(y)| \le M\|f\|_u$$

for all n. The constant function $M\|f\|_u$ is integrable on $[0,1]$, so we can apply the Dominated Convergence Theorem to conclude that

$$\int_0^1 k(x_n, y)f(y)\, dy \to \int_0^1 k(p, y)f(y)\, dy,$$

i.e., $(Tf)(x_n) \to (Tf)(p)$. Since the sequence was arbitrary, we conclude that Tf is continuous at p. Since p was arbitrary, we conclude that Tf is continuous.

Thus we have an operator $T\colon C([0,1]) \to C([0,1)$. It's straightforward to prove T is linear: if $f, g \in C([0,1])$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned}
[T(\alpha f + \beta g)](x) &= \int_0^1 k(x,y)(\alpha f + \beta g)(y)\, dy \\
&= \int_0^1 k(x,y)(\alpha f(y) + \beta g(y))\, dy \\
&= \int_0^1 [\alpha k(x,y)f(y) + \beta k(x,y)g(y)]\, dy \\
&= \alpha \int_0^1 k(x,y)f(y)\, dy + \beta \int_0^1 k(x,y)g(y)\, dy \\
&= \alpha(Tf)(x) + \beta(Tg)(x) \\
&= [\alpha Tf + \beta Tg](x).
\end{aligned}$$

Thus, $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$, so T is linear.

Next we want to show that T is a bounded linear operator, i.e., there is a constant C so that $\|Tf\|_u \leq C\|f\|_u$ for all $f \in C([0,1])$. We have

$$\begin{aligned}
|(Tf)(x)| &= \left|\int_0^1 k(x,y)f(y)\,dy\right| \\
&\leq \int_0^1 |k(x,y)||f(y)|\,dy \\
&\leq \int_0^1 M\|f\|_u\,dy \\
&= (1-0)M\|f\|_u \\
&= M\|f\|_u,
\end{aligned}$$

Thus, we have $|(Tf)(x)| \leq M\|f\|_u$ for all x. Taking the sup over x, we conclude $\|Tf\|_u \leq M\|f\|_u$ (so we can take $C = M$).

Finally, we want to apply the Arzelà-Ascoli Theorem to show that $T(B)$ is pre-compact. According to the Arzelà-Ascoli Theorem, a subset of $C([0,1])$ is pre-compact if it is pointwise bounded and equicontinuous. We want to apply this to $T(B)$.

It's easy to see that $T(B)$ is pointwise bounded, since for every $x \in [0,1]$ and $f \in B$, we have

$$|(Tf)(x)| \leq \|Tf\|_u \leq M\|f\|_u \leq M,$$

since $\|f\|_u \leq 1$ for $f \in B$. (In fact, we see that $T(B)$ is uniformly bounded.)

To see that $T(B)$ is equicontinuous at each point, fix $p \in [0,1]$. We need to show that given $\varepsilon > 0$, there is a $\delta > 0$ so that $|(Tf)(x) - (Tf)(p)| < \varepsilon$ whenever $|x-p| < \delta$ and $f \in B$.

To prove this, note that for $f \in B$,

$$\begin{aligned}
|(Tf)(x) - (Tf)(p)| &= \left|\int_0^1 k(x,y)f(y)\,dy - \int_0^1 k(p,y)f(y)\,dy\right| \\
&= \left|\int_0^1 [k(x,y) - k(p,y)]f(y)\,dy\right| \\
&\leq \int_0^1 |k(x,y) - k(p,y)||f(y)|\,dy \\
&\leq \int_0^1 |k(x,y) - k(p,y)|\|f\|_u\,dy \\
&= \|f\|_u \int_0^1 |k(x,y) - k(p,y)|\,dy \\
&\leq \int_0^1 |k(x,y) - k(p,y)|\,dy.
\end{aligned}$$

For a fixed y, $k(x,y) - k(p,y) \to 0$ as $x \to p$, by the continuity of k. We also have $|k(x,y) - k(p,y)| \leq 2M$ and the constant function $2M$ is integrable on $[0,1]$, so

$$\lim_{x\to p}\int_0^1 |k(x,y)-k(p,y)|\,dy = 0$$

by the Dominated Convergence Theorem (as we often do, I'm skipping the part about reducing to a sequence of x's). Thus, given $\varepsilon > 0$, we can find $\delta > 0$ so that

$$\int_0^1 |k(x,y)-k(p,y)|\,dy < \varepsilon$$

for $|x-p| < \delta$. But then we have $|(Tf)(x) - (Tf)(p)| < \varepsilon$ when $|x-p| < \varepsilon$ and $f \in B$. This completes the proof.

Remark. Some people tried to prove the continuity of Tf and some of the other parts of the problem using the Mean Value Theorem for Integrals. I suppose that the relevant theorem is this.

Mean Value Theorem For Integrals. *Let $f, h\colon [a,b] \to \mathbb{R}$ be continuous and suppose that f has constant sign (i.e., $f \geq 0$ or $f \leq 0$). Then there is some $c \in [a,b]$ such that*

$$\int_a^b h(y)f(y)\,dy = h(c)\int_a^b f(y)\,dy$$

The argument people gave for the continuity of Tf was something like this: By the mean value theorem for integrals, there is a constant c such that

$$(Tf)(x) = \int_0^1 k(x,y)f(y)\,dy = k(x,c)\int_0^1 f(y)\,dy.$$

The integral $\int_0^1 f(y)\,dy$ is just a constant and $x \mapsto k(x,c)$ is continuous, so $x \mapsto (Tf)(x)$ is continuous.

Now there are two problems with this. The first is just a matter of detail: the Mean Value Theorem assumes that the functions are real and that f has constant sign, whereas we are assuming k and f are complex valued. This problem can be overcome by writhing $k = k_1 + ik_2$ where k_1, k_2 are real, and $f = f_1 + if_2 = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$, where f_1 and f_2 are real. Plugging this into the definition of Tf and working out the algebra would give Tf as a linear combination of integrals to which the Mean Value Theorem could be applied, and it would suffice to show that each of these integrals defines a continuous function.

Unfortunately, the second problem is more serious. Suppose that we have reduced the problem, as above, to the case

$$(Tf)(x) = \int_0^1 k(x,y)f(y)\,dy$$

where k is real-valued and $f \geq 0$. We can apply the Mean Value Theorem, but what is the result? For each fixed x, we are going to apply the Mean Value Theorem with $h(y) = k(x,y)$. For each x, we have a *different* function h, so we get a *different* constant c; we have to expect that c will change when x changes, so what we really have is

$$(Tf)(x) = k(x, c_x) \int_0^1 f(y)\, dy.$$

with constants c_x depending on x. The statement of the Mean Value Theorem doesn't tell us anything about how c_x depends on x. (Is $x \mapsto c_x$ continuous? The Theorem doesn't say.) Without any information about the function $x \mapsto c_x$, there is no reason to suppose that $x \mapsto k(x, c_x)$ is continuous.

I don't see that the Mean Value Theorem for Integrals is any help in proving Tf is continuous. There are two approaches I can see that would work: Use the Dominated Convergence Theorem as above, or, for a Baby Reals proof, use the fact that k is *uniformly* continuous, because $[0, 1] \times [0, 1]$ is compact.

Problem 4. Let $G\colon \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function. Let μ_G be the Lebesgue-Stieljes measure associated to G. Suppose that G is absolutely continuous. Show that

$$\int f \, d\mu_G = \int f G' \, dm$$

for all Borel measurable functions $f\colon \mathbb{R} \to [0, \infty]$.

Let f be Borel measurable. Show that $f \in L^1(\mu_G)$ if and only in $fG' \in L^1(m)$ and in this case

$$\int f \, d\mu_G = \int f G' \, dm.$$

Answer:
Since G is absolutely continuous, we have $\mu_G \ll m$. By the Radon-Nikodym theorem, there is a nonnegative Borel measurable function h so that

$$\mu_G(E) = \int_E h \, dm$$

for all Borel sets $E \subseteq \mathbb{R}$. Since $\mu_G \ll m$, both sides of this equation are 0 if E is a Lebesgue null set, so we can extend this equation to all Lebesgue measurable sets (see p. 35). By the Lebesgue Differentiation Theorem quoted earlier, for almost all x,

$$h(x) = \lim_{r \to 0} \frac{1}{\mu(E_r)} \int_{E_r} h \, dm$$

if $\{ E_r \}_{r>0}$ is a collection of sets that shrinks "nicely" to x. This means that $E_r \subseteq \mathrm{B}(r, x)$ and that there is some $\alpha > 0$ independent of r so that $m(E_r) > \alpha m(\mathrm{B}(r, x))$.

To apply this theorem, let $E_r = (x, x + r/2] \subseteq (x - r, x + r) = \mathrm{B}(r, x)$. We have $m(E_r) = r/2 = \frac{1}{4}(2r) = \frac{1}{4}m(\mathrm{B}(r, x))$, so we can take α to be any positive number less than 1/4. Thus, E_r shrinks nicely to x. Then, for almost all x,

$$\begin{aligned} h(x) &= \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} h\, dm \\ &= \lim_{r\to 0} \frac{1}{r/2} \mu_G((x, x + r/2]) \\ &= \lim_{r\to 0} \frac{G(x + r/2) - G(x)}{\frac{r}{2}}. \end{aligned}$$

Since we're only considering $r > 0$ here, we conclude that G has a right-hand derivative at almost all points, and h is equal to the right-hand derivative a.e. By considering the sets $(x - r/2, x]$, we can conclude by a similar argument that G has a left-hand derivative a.e. Thus, G is differentiable a.e. and $h = G'$ a.e. Thus, we have

$$\mu_G(E) = \int_E G'\, dm$$

for all Lebesgue measurable sets E.

If $\varphi = \sum_{j=1}^n a_j \chi_{E_j}$ is a simple function, we have

$$\begin{aligned} \int \varphi\, d\mu_G &= \sum_{j=1}^n a_j \int \chi_{E_j}\, d\mu_G \\ &= \sum_{j=1}^n a_j \mu_G(E_j) \\ &= \sum_{j=1}^n a_j \int_{E_j} G'\, dm \\ &= \sum_{j=1}^n a_j \int \chi_{E_j} G'\, dm \\ &= \int \varphi G'\, dm, \end{aligned}$$

Thus, we have

$$\int \varphi\, d\mu_G = \int \varphi G'\, dm,$$

for all simple functions. If $f\colon \mathbb{R} \to [0, \infty]$ is measurable, we can find simple functions φ_n that increase pointwise to f. From the above, we have

$$(*) \qquad \int \varphi_n\, d\mu_G = \int \varphi_n G'\, dm$$

for all n. Since $G' \geq 0$ (G is increasing), the functions $\varphi_n G'$ increase pointwise to fG'. Thus, letting $n \to \infty$ in $(*)$, we get

$$(**) \qquad \int f\, d\mu_G = \int fG'\, dm$$

for all nonnegative measurable functions f. If f is a complex-valued function, $f \in L^1(\mu_G)$ if and only if

$$\int |f| \, d\mu_G < \infty$$

From $(**)$, this is equivalent to

$$\int |f| G' \, d\mu < \infty.$$

Since $G' \geq 0$, this is equivalent to $fG' \in L^1(m)$. Thus, $f \in L^1(\mu_G)$ if and only if $fG' \in L^1(m)$. If $f \in L^1(\mu_G)$ is real-valued, we can write $f = f^+ - f^-$. By $(**)$, we have

$$\int f^\pm \, d\mu_G = \int f^\pm G' \, dm,$$

where both sides are finite. Subtracting these two equations gives

$$\int f \, d\mu_G = \int fG' \, dm$$

for all real-valued functions $f \in L^1(\mu_G)$. If $f \in L^1(\mu_G)$ is complex-valued, we can apply this equation to the real and imaginary parts of f to conclude that the equation holds for all functions in $L^1(\mu_G)$.

Problem 5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f \in L^p(X)$ and $\{ f_n \}_{n=1}^{\infty} \subseteq L^p(X)$, where $1 < p < \infty$.

Show that if $f_n \to f$ in L^p, then $f_n \to f$ in measure. Remark: Hence there is a subsequence that converges to f a.e.

Answer:
Suppose that $f_n \to f$ in L^p, i.e., $\|f_n - f\|_p \to 0$ as $n \to \infty$.

To show that $f_n \to f$ in measure, we must show that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{ x \mid |f_n(x) - f(x)| \geq \varepsilon \}) = 0.$$

To do, this let $\varepsilon > 0$ be arbitrary and define

$$E_n = \{ x \mid |f_n(x) - f(x)| \geq \varepsilon \}.$$

On E_n, we have $\varepsilon \leq |f_n(x) - f(x)|$, and so $\varepsilon^p \leq |f_n(x) - f(x)|^p$. Thus, on X we have

$$\varepsilon^p \chi_{E_n} \leq \chi_{E_n} |f_n - f|^p \leq |f_n - f|^p.$$

Integrating this inequality, we get

$$\varepsilon^p \mu(E_n) \leq \|f_n - f\|_p^p$$

and hence

$$\mu(E_n) \leq \frac{1}{\varepsilon^p}\|f_n - f\|_p^p.$$

Since the right-hand side goes to 0 as $n \to \infty$, we conclude that $\mu(E_n) \to 0$. Since ε was arbitrary, the proof is complete.

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose that $1 \leq r < s \leq \infty$. Show that $L^s(X) \subseteq L^r(X)$ and the inclusion mapping $L^s(X) \hookrightarrow L^r(X)$ is bounded. Hints: $|f|^r = |f|^r \cdot 1$; s/r and $s/(s-r)$ are conjugate exponents.

Answer:
Recall Hölder's inequality: For measurable functions $f, g\colon X \to \mathbb{C}$ and conjugate exponents p and q, we have

$$\int |fg|\, d\mu \leq \|f\|_p \|g\|_q$$

To solve the problem, first suppose that $1 \leq r < s < \infty$ and that $f \in L^s$. By Hölder's inequality for the conjugate exponents s/r and $s/(s-r)$, we have

$$\begin{aligned}
\|f\|_r^r &= \int |f|^r\, d\mu \\
&= \int |f|^r \cdot 1\, d\mu \\
&\leq \||f|^r\|_{s/r} \|1\|_{s/(s-r)}.
\end{aligned}$$

We now compute the two norms on the right in this inequality. First, we have

$$\begin{aligned}
\||f|^r\|_{s/r}^{s/r} &= \int \big||f|^r\big|^{s/r}\, d\mu \\
&= \int |f|^s\, d\mu \\
&= \|f\|_s^s,
\end{aligned}$$

and hence

$$\||f|^r\|_{s/r} = \{\, \|f\|_s^s \,\}^{r/s} = \|f\|_s^r.$$

We also have

$$\begin{aligned}
\|1\|_{s/(s-r)}^{s/(s-r)} &= \int 1^{s/(s-r)}\, d\mu \\
&= \int 1\, d\mu \\
&= \mu(X),
\end{aligned}$$

and so

$$\|1\|_{s/(s-r)} = \mu(X)^{(s-r)/s}.$$

Plugging these computations back into the result of Hölder's inequality, we get

$$\|f\|_r^r \leq \mu(X)^{(s-r)/s}\|f\|_s^r$$

and so by taking roots we get

$$\|f\|_r \leq \mu(X)^{(s-r)/(rs)}\|f\|_s$$

and so finally

$$\|f\|_r \leq \mu(X)^{1/r-1/s}\|f\|_s.$$

Since $f \in L^s$, the right-hand side is finite, so $\|f\|_r < \infty$, which means that $f \in L^r$. If $j\colon L^s \hookrightarrow L^r$ is the inclusion map, we see that

$$\|j(f)\|_r = \|f\|_r \leq C\|f\|_s$$

(where $C = \mu(X)^{1/r-1/s}$) so j is a bounded (i.e., continuous) linear map.

The case $1 \leq r < s = \infty$ is even easier. Since $|f| \leq \|f\|_\infty$ a.e., we have

$$\begin{aligned}
\|f\|_r^r &= \int |f|^r \, d\mu \\
&\leq \int \|f\|_\infty^r \, d\mu \\
&= \|f\|_\infty^r \int 1 \, d\mu \\
&= \|f\|_\infty^r \mu(X),
\end{aligned}$$

and thus

$$\|f\|_r \leq \mu(X)^{1/r}\|f\|_\infty.$$

As above, this inequality shows that $f \in L^r$ and that the inclusion $L^\infty \hookrightarrow L^r$ is continuous.