Problem Set

Problem Set #1

Math 5322, Fall 2001

March 4, 2002

ANSWERS

All of the problems are from Chapter 3 of the text.

Problem 1. [Problem 2, page 88]

If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

Answer:
For the first part of the problem, let $X = P \cup N$ be a Hahn decomposition of X with respect to ν. Thus, we have

$$\nu^+(E) = \nu(E \cap P)$$
$$\nu^-(E) = -\nu(E \cap N)$$
$$|\nu| = \nu^+ + \nu^-$$

Assume that E is ν-null. This means that for all measurable $F \subseteq E$, $\nu(F) = 0$. But then $E \cap P \subseteq E$, so $\nu^+(E) = 0$ and $E \cap N \subseteq E$, so $\nu^-(E) = 0$. Then we have $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$

Conversely, suppose that $|\nu|(E) = 0$. Let $F \subseteq E$ be measurable. Then $0 \leq \nu^+(F) + \nu^-(F) = |\nu|(F) \leq |\nu|(E) = 0$, so $\nu^+(F) = 0$ and $\nu^-(F) = 0$. But then $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. Thus, we can conclude that E is ν-null.

For the second part of the problem, we want to show that the following conditions are equivalent.

(1) $\nu \perp \mu$

(2) $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

(3) $|\nu| \perp \mu$

Let's first show that (1) $\implies$ (2). Since $\nu \perp \mu$, we can decompose X into a disjoint union of measurable sets A and B so that A is μ-null and B is ν-null. We can also find an Hahn decomposition $X = P \cup N$ with respect to ν, as above. Since B in ν-null, we have $\nu^+(B) = \nu(P \cap B) = 0$ and $\nu^-(B) = -\nu(B \cap N) = 0$. Thus, B is ν^+-null and ν^--null. Since A is still μ-null, we conclude that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Next, we show that (2) $\implies$ (3). Since $\nu^+ \perp \mu$, we can find disjoint measurable sets A_1 and B_1 so that $X = A_1 \cup B_1$, $\nu^+(B_1) = 0$ and A_1 is μ-null. Similarly, we can write $X = A_2 \cup B_2$ (disjoint union) where $\nu^-(B_2) = 0$ and A_2 is μ-null. We can then partition X as

$$X = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (B_1 \cap B_2)$$

Now, $(A_1 \cap A_2)$ is contained in the μ-null set A_1, and so is μ-null. Similarly, $A_2 \cap B_2$ and $A_2 \cap B_1$ are μ-null. We have $0 \leq \nu^+(B_1 \cap B_2) \leq \nu^+(B_1) = 0$, so $\nu^+(B_1 \cap B_2) = 0$. Similarly, $\nu^-(B_1 \cap B_2) = 0$, and thus $|\nu|(B_1 \cap B_2) = 0$. We thus have a disjoint measurable decomposition $X = A_3 \cup B_3$ where

$$A_3 = (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1)$$
$$B_3 = B_1 \cap B_2.$$

The set A_3 is μ-null and B_3 is $|\nu|$-null. This shows that $|\nu| \perp \mu$.

Finally we show that (3) $\implies$ (1). So, suppose that $|\nu| \perp \mu$. The we have a measurable decomposition $X = A \cup B$ where $|\nu|(B) = 0$ and A is μ-null. However, $|\nu|(B) = 0$ implies that B is ν-null (as we proved above), so we have $\nu \perp \mu$.

Problem 2. [Problem 3, page 88]

Let ν be a signed measure on $(X, \mathcal{M})$.

a. $L^1(\nu) = L^1(|\nu|)$.

b. If $f \in L^1(\nu)$,

$$\left| \int f \, d\nu \right| \leq \int |f| \, d|\nu|.$$

c. If $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \left\{ \left| \int_E f \, d\nu \right| \mid |f| \leq 1 \right\}$$

Answer:

According to the definition on page 88 of the text, the definition of $L^1(\nu)$ is $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and if $f \in L^1(\nu)$, we define

$$\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-.$$

So, consider part (a.) of the problem. If $f \in L^1(\nu)$, then, by definition the integrals

$$(*) \qquad \int |f| \, d\nu^+, \qquad \int |f| \, d\nu^-$$

are finite. On the other hand, $|\nu|$ is defined by $|\nu|(E) = \nu^+(E) + \nu^-(E)$. Hence, the equation

$$\int \chi_E \, d|\nu| = \int \chi_E \, d\nu^+ + \int \chi_E \, d\nu^-$$

holds by definition. Thus, we can apply the usual argument to show that

$$(**) \qquad \int g \, d|\nu| = \int g \, d\nu^+ + \int g \, d\nu^-$$

holds for all measurable $g \colon X \to [0, \infty]$: This equations holds for characteristic functions, hence (by linearity) for simple functions and hence (by the monotone convergence theorem) for nonnegative functions. Thus, if the integrals in $(*)$ are finite, the integral $\int |f| \, d|\nu|$ is finite, so $f \in L^1(|\nu|)$.

Conversely, if $f \in L^1(|\nu|)$, the integral $\int |f| \, d|\nu|$ is finite, and so by $(**)$, the integrals in $(*)$ are finite, so $f \in L^1(\nu)$.

Next consider part (b.) of the problem. If $f \in L^1(\nu)$, then by definition

$$\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^- = \int f^+\,d\nu^+ - \int f^-\,d\nu^+ - \int f^+\,d\nu^- + \int f^-\,d\nu^-,$$

where all of the integrals on the right are positive numbers. Hence, by the triangle inequality,

$$\begin{aligned} \left|\int f\,d\nu\right| &\le \int f^+\,d\nu^+ + \int f^-\,d\nu^+ + \int f^+\,d\nu^- + \int f^-\,d\nu^- \\ &= \int (f^+ + f^-)\,d\nu^+ + \int (f^+ + f^-)\,d\nu^- \\ &= \int |f|\,d\nu^+ + \int |f|\,d\nu^- \\ &= \int |f|\,d|\nu| \end{aligned}$$

Finally, consider part (c.) of the problem. If $|f| \le 1$, then for any $E \in \mathcal{M}$, $|f|\chi_E \le \chi_E$. Thus, we have

$$\begin{aligned} \left|\int_E f\,d\nu\right| &\le \int_E |f|\,d|\nu| \\ &= \int_X |f|\chi_E\,d|\nu| \\ &\le \int_X \chi_E\,d|\nu| \\ &= |\nu|(E). \end{aligned}$$

Thus, the sup in part (c.) is $\le |\nu|(E)$.

To get the reverse inequality, let $X = P \cup N$ be a Hahn decomposition of X with respect to ν, and recall the description of ν^+ and ν^- in terms of P and N from the last problem. Then we have

$$|\nu|(E) = \nu(E \cap P) - \nu(E \cap N) = \int (\chi_{E\cap P} - \chi_{E\cap N})\,d\nu.$$

But, $|\chi_{E\cap P} - \chi_{E\cap N}| \le 1$ (P and N are disjoint). This shows that $|\nu|(E)$ is an element of the set we're supping over, so $|\nu|(E) \le$ the sup.

Problem 3. **[Problem 4, page 88]**

If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.

Answer:

Let $X = P \cup N$ be a Hahn decomposition of X with respect to ν. Then for any measurable set E, we have

$$(*) \qquad \nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P).$$

If $\nu^+(E) = \infty$, this equation shows $\lambda(E \cap P) = \infty$, and so $\lambda(E) = \infty$. Thus, $\nu^+(E) \geq \lambda(E)$ is true in this case. If $\nu^+(E) < \infty$, then both numbers on the right of $(*)$ must be finite, and $(*)$ gives $\nu^+(E) + \mu(E \cap P) = \lambda(E \cap P)$. Thus, we have

$$\nu^+(E) \leq \nu^+(E) + \mu(E \cap P) = \lambda(E \cap P) \leq \lambda(E).$$

The argument for the other inequality is similar. We have

$$\nu^-(E) = -\nu(E \cap N) = -\lambda(E \cap N) + \mu(E \cap N)$$

If $\nu^-(E) = \infty$, then we must have $\mu(E) \geq \mu(E \cap N) = \infty$, so $\mu(E) \geq \nu^-(E)$. Otherwise, every thing must be finite and

$$\nu^-(E) \leq \nu^-(E) + \lambda(E \cap N) = \mu(E \cap N) \leq \mu(E).$$

Problem 4. [Problem 5, page 88] If ν_1 and ν_2 are signed measures that both omitted the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Use Exercise 4.)

Answer:
The hypothesis that $\pm\infty$ is omitted assures that the signed measure $\nu_1 + \nu_2$ is defined.

We have $v_1 = \nu_1^+ - \nu_1^-$ and $\nu_2 = \nu_2^+ - \nu_2^-$. Adding these equations gives

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-),$$

where the expressions in parentheses are positive measures. Thus, by Exercise 4, we have

$$(\nu_1 + \nu_2)^+ \leq \nu_1^+ + \nu_2^+$$
$$(\nu_1 + \nu_2)^- \leq \nu_1^- + \nu_2^-$$

and adding these inequalities gives $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Problem 5. [Problem 12, page 92]

For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Answer:
We first want to show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. So, suppose that $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ and $(\mu_1 \times \mu_2)(E) = 0$. Recall that the x_1-section of E is defined by

$$E_{x_1} = \{ x_2 \in X_2 \mid (x_1, x_2) \in E \}.$$

By Theorem 2.36 on page 66 of the text, the function $x_1 \mapsto \mu_2(E_{x_1})$ is measurable and

$$0 = (\mu_1 \times \mu_2)(E) = \int_{X_1} \mu_2(E_{x_1})\, d\mu_1(x_1)$$

Thus, $\mu_2(E_{x_1}) = 0$ for μ_1-almost all x_1. In other words, there is a μ_1-null set $N \subseteq X_1$ so that $\mu_2(E_{x_1}) = 0$ if $x_1 \notin N$. If $x_1 \notin N$, we must have $\nu_2(E_{x_1}) = 0$, since $\nu_2 \ll \mu_2$. We also have $\nu_1(N) = 0$, since $\nu_1 \ll \mu_1$. Thus, $x_1 \mapsto \nu_2(E_{x_1})$ is zero almost everywhere with respect to ν_1. Thus, we have

$$(\nu_1 \times \nu_2)(E) = \int_{X_1} \nu_2(E_{x_1})\, d\nu_1(x_1) = 0.$$

Thus, $(\mu_1 \times \mu_2)(E) = 0 \implies (\nu_1 \times \nu_2)(E) = 0$, so $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

For notational simplicity, let

$$f = \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}, \qquad f_1 = \frac{d\nu_1}{d\mu_1}, \qquad f_2 = \frac{d\nu_2}{d\mu_2}.$$

The, we have

$$(*) \qquad (\nu_1 \times \nu_2)(E) = \int_E f(x_1, x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2)$$

and f is the unique (up to equality almost everywhere) function with this property. Of course we have

$$\nu_1(A) = \int_A f_1(x_1)\, d\mu_1(x_1)$$
$$\nu_2(B) = \int_B f_2(x_2)\, d\mu_2(x_2)$$

if $A \subseteq X_1$ and $B \subseteq X_2$ are measurable.

On the other hand, the functions $(x_1, x_2) \mapsto f_1(x_1)$ and $(x_1, x_2) \mapsto f_2(x_2)$ are measurable, so we can define a measure λ on $X_1 \times X_2$ by

$$(**) \qquad \lambda(E) = \int_E f_1(x_1) f_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2).$$

Suppose that $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$. Then, we can use Tonelli's Theorem to calculate $\lambda(A_1 \times A_2)$ as follows:

$$\begin{aligned}
\lambda(A_1 \times A_2) &= \int_{A_1 \times A_2} f_1(x_1) f_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) \\
&= \int_{X_1 \times X_2} \chi_{A_1 \times A_2}(x_1, x_2) f_1(x_1) f_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) \\
&= \int_{X_1 \times X_2} \chi_{A_1}(x_1) \chi_{A_2}(x_2) f_1(x_1) f_2(x_2)\, d(\mu_1 \times \mu_2)(x_1, x_2) \\
&= \int_{X_1} \int_{X_2} \chi_{A_1}(x_1) \chi_{A_2}(x_2) f_1(x_1) f_2(x_2)\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&= \int_{X_1} \chi_{A_1}(x_1) f_1(x_1) \int_{X_2} \chi_{A_2}(x_2) f_2(x_2)\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&= \int_{X_1} \chi_{A_1}(x_1) f_1(x_1) \nu_2(A_2)\, d\mu_1(x_1) \\
&= \nu_2(A_2) \int_{X_1} \chi_{A_1}(x_1) f_1(x_1)\, d\mu_1(x_1) \\
&= \nu_1(A_1) \nu_2(A_2).
\end{aligned}$$

Thus we have

$$\lambda(A_1 \times A_2) = \nu_1(A_1)\nu_2(A_2) = (\nu_1 \times \nu_2)(A_1 \times A_2).$$

for all measurable rectangles $A_1 \times A_2$. Since our measures are σ-finite, this shows that $\lambda = \nu_1 \times \nu_2$, see the remark at the bottom of page 64 in the text. Comparing $(*)$ and $(**)$ and using the uniqueness in $(*)$ then shows that $f(x_1, x_2) = f_1(x_1) f_2(x_2)$

Problem 6. **[Problem 16, page 92]** Suppose that μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then $0 \leq f < 1$ μ-a.e. and $d\nu/d\mu = f/(1 - f)$.

Answer:
It's clear that $\nu \ll \lambda$, so $f = d\nu/d\lambda$ makes sense.

We want to show $0 \leq f < 1$ μ-a.e. Suppose, for a contradiction, that the is a set E with $\mu(E) > 0$ and $f \geq 1$ on E. On the on hand, we have $\nu(E) < \nu(E) + \mu(E) = \lambda(E)$. On the other hand, $f\chi_E \geq \chi_E$, so

$$\nu(E) = \int_E f\, d\lambda = \int \chi_E f\, d\lambda \geq \int \chi_E\, d\lambda = \lambda(E),$$

a contradiction.

If E is measurable, we have

$$\int_E 1\, d\nu = \nu(E) = \int_E f\, d\lambda = \int_E f\, d\mu + \int_E f\, d\nu,$$

which yields

$$\int_E (1-f)\, d\nu = \int_E f\, d\mu,$$

or, in other words,

$$\int \chi_E (1-f)\, d\nu = \int \chi_E f\, d\mu.$$

By the usual procedure, we can extend this equation from characteristic functions to nonnegative functions. Thus, we have

$$\int g(1-f)\, d\nu = \int gf\, d\mu$$

for all nonnegative measurable functions g. Hence, for any measurable set E, we have

$$(*) \qquad \int_E g(1-f)\, d\nu = \int_E gf\, d\mu$$

for all nonnegative measurable functions g.

Since $0 \leq f < 1$ μ-a.e. the function $1/(1-f)$ is defined and nonnegative μ-a.e. Since $\nu \ll \mu$, $1/(1-f)$ is also defined and nonnegative ν-a.e. Thus, we can set $g = 1/(1-f)$ in $(*)$. This gives

$$\nu(E) = \int_E \frac{f}{1-f}\, d\mu,$$

for all measurable E, whence $d\nu/d\mu = f/(1-f)$.

Problem 7. **[Problem 18, page 94]** Prove Proposition 3.13c.

In other words, suppose that ν is a complex measure on $(X, \mathcal{M})$. Then $L^1(\nu) = L^1(|\nu|)$ and if $f \in L^1(\nu)$,

$$\left| \int f\, d\nu \right| \leq \int |f|\, d|\nu|.$$

Answer:

I seem to have been very confused the day I tried to do this in class!

Let $z = x + iy$ be a complex number, where x, y are real. We have

$$x^2, y^2 \leq x^2 + y^2 = |z|^2$$

so taking square roots gives $|x|, |y| \leq |z|$. One the other hand, we have

$$|z|^2 = x^2 + y^2 = |x|^2 + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = [|x| + |y|]^2,$$

and so $|z| \leq |x| + |y|$.

Now lets begin the solution of the problem. By the definition on page 93 of the text, $L^1(\nu)$ is defined to be $L^1(\nu_r) \cap L^1(\nu_i)$ and for $f \in L^1(\nu)$, we define

$$\int f\, d\nu = \int f\, d\nu_r + i \int f\, d\nu_i.$$

On the other hand, we know from Exercise 3 on page 88 of the text (one of the problems in this set) that for a signed measure λ, $L^1(\lambda) = L^1(|\lambda|)$ and for $f \in L^1(\lambda)$,

$$\left| \int f\, d\lambda \right| \leq \int |f|\, d|\lambda|.$$

Hence in our present situation, we know $L^1(\nu) = L^1(|\nu_r|) \cap L^1(|\nu_i|)$.

Let's recall the reasoning of part b. of Proposition 3.13. We can find some finite positive measure μ such that ν_r and ν_i are absolutely continuous with respect to μ ($\mu = |\nu_r| + |\nu_i|$ will do). Then, by the Lebesgue-Radon-Nikodym Theorem, there are real-valued measurable functions f and g such that $d\nu_r = f\, d\mu$ and $d\nu_i = g\, d\mu$. Then, we have

$$(*) \qquad \nu(E) = \int_E h\, d\mu$$

where $h = f + ig$. From this equation, we have $d|\nu| = |h|\, d\mu$, by definition. Since $|f|, |g| \leq |h|$ we have

$$\begin{aligned} |\nu_r|(E) &= \int_E |f|\, d\mu \leq \int_E |h|\, d\mu = |\nu|(E) \\ |\nu_i|(E) &= \int_E |g|\, d\mu \leq \int_E |h|\, d\mu = |\nu|(E) \end{aligned}$$

so we certainly have $\nu_r, \nu_i \ll |\nu|$. Thus, once again, we can find real-valued functions φ and η such that $d\nu_r = \varphi\, d|\nu|$ and $d\nu_i = \eta\, d|\nu|$. If $\varphi = \psi + i\eta$, then

$$\nu(E) = \int_E \varphi\, d|\nu|.$$

But then, comparing with $(*)$, we have

$$\int_E \varphi |h|\, d\mu = \int_E h\, d\mu.$$

By the uniqueness part of the Lebesgue-Radon-Nikodym Theorem, $\varphi|h| = h$, μ-a.e., and hence $|\nu|$-a.e. On the other hand if Z is the set where $h = 0$, then

$$|\nu|(Z) = \int_Z |h|\, d\mu = \int_Z 0\, d\mu = 0$$

so $h \neq 0$, $|\nu|$-a.e. This shows that $|\varphi| = 1$, $|\nu|$-a.e., so we can conclude that $|\psi|, |\eta| \leq |\varphi| = 1$, $|\nu|$-a.e., and $1 = |\varphi| \leq |\psi| + |\eta|$, $|\nu|$-a.e. Finally, we have $d|\nu_r| = |\psi|\, d|\nu|$ and $d|\nu_i| = |\eta|\, d|\nu|$.

Now, suppose that $f \in L^1(|\nu|)$. Then

$$\int |f| \, d|\nu| < \infty$$

Since $|f||\psi| \leq |f|$, $|\nu|$-a.e., we have

$$\int |f||\psi| \, d|\nu| \leq \int |f| \, d|\nu| < \infty$$

but the integral on the left is

$$\int |f| \, d|\nu_r|,$$

so $f \in L^1(|\nu_r|)$. Similarly, $f \in L^1(|\nu_i|)$

Conversely, suppose that $f \in L^1(|\nu_r|) \cap L^1(|\nu_i|)$ Then we have

$$\int |f||\psi| \, d|\nu| = \int |f| \, d|\nu_r| < \infty$$
$$\int |f||\eta| \, d|\nu| = \int |f| \, d|\nu_i| < \infty.$$

and so we have

$$\int |f|[|\psi| + |\eta|] \, d|\nu| < \infty.$$

However, we have $1 \leq |\psi| + |\eta|$, $|\nu|$-a.e., so

$$\int |f| \, d|\nu| \leq \int |f|[|\psi| + |\eta|] \, d|\nu| < \infty.$$

and so $f \in L^1(|\nu|)$.

Finally, suppose that $f \in L^1(\nu) = L^1(|\nu|)$, we then have

$$\begin{aligned}
\left| \int f \, d\nu \right| &= \left| \int f \, d\nu_r + i \int f \, d\nu_i \right| && \text{by definition} \\
&= \left| \int f\psi \, d|\nu| + i \int f\eta \, d|\nu| \right| \\
&= \left| \int f[\psi + i\eta] \, d|\nu| \right| \\
&= \left| \int f\varphi \, d|\nu| \right| \\
&\leq \int |f||\varphi| \, d|\nu| \\
&= \int |f| \, d|\nu|,
\end{aligned}$$

since $|\varphi| = 1$, $|\nu|$-a.e. This completes the proof.

Problem 8. **[Problem 20, page 94]** If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Answer:
Let $f = d\nu/d|\nu|$, so we have

$$(*) \qquad \nu(E) = \int_E f\, d|\nu|$$

for all measurable sets E, and we know from Proposition 3.13 on page 94 of the text that $|f| = 1$ $|\nu|$-a.e. We may as well suppose that $|f| = 1$ everywhere.

We can write the complex function f as $f = g + ih$, where g and h are real-valued. We can also write $\nu = \nu_r + i\nu_i$ for finite signed measures ν_r and ν_i. Thus, we have

$$\nu_r(E) + i\nu_i(E) = \nu(E) = \int_E f\, d|\nu| = \int_E g\, d|\nu| + i \int_E h\, d|\nu|.$$

Comparing real and imaginary parts, we get

$$\nu_r(E) = \int_E g\, d|\nu|$$
$$\nu_i(E) = \int_E h\, d|\nu|.$$

By hypothesis, we have $\nu(X) = |\nu|(X)$, so

$$\int_X g\, d|\nu| + i \int_X h\, d|\nu| = |\nu|(X).$$

Since the right-hand side is real, the imaginary part of the left-hand side must be zero, and we have

$$\int_X g\, d|\nu| = |\nu|(X) = \int_X 1\, d|\nu|,$$

and so

$$(**) \qquad 0 = \int_X (1 - g)\, d|\nu|.$$

Since g is the real part of f, we have

$$g \le |g| \le |g + ih| = |f| = 1$$

so $1 - g \ge 0$. But then $(**)$ shows that $1 - g = 0$ a.e., so $g = 1$ a.e.

We then have $1 = |f|^2 = g^2 + h^2 = 1 + h^2$ a.e., so $h = 0$ a.e. and hence $f = g = 1$ a.e. Putting this in $(*)$ shows that $\nu = |\nu|$.

Problem 9. **[Problem 20, page 94]** Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define

$$\text{(A)} \quad \mu_1(E) = \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| \mid n \in \mathbb{N},\ E_1, \ldots, E_n \text{ disjoint, } E = \bigcup_{j=1}^{n} E_j \right\},$$

$$\text{(B)} \quad \mu_2(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| \mid E_1, E_2, \ldots \text{ disjoint, } E = \bigcup_{j=1}^{\infty} E_j \right\},$$

$$\text{(C)} \quad \mu_3(E) = \sup \left\{ \left| \int_E f\, d\nu \right| \mid |f| \le 1 \right\}.$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$.

Answer:
Although the author didn't explicitly say so, the sets E_j in (A) and (B) should, of course, be measurable and the functions f in (C) should be measurable. Also note that the book has a typo in (C) ($d\mu$ instead of $d\nu$).

As suggested in the book, we first prove that $\mu_1 \le \mu_2 \le \mu_3 \le \mu_1$ and then that $\mu_3 = |\nu|$.

We have $\mu_1 \le \mu_2$, because the sums in (A) are among the sums in (B): given a finite sequence $E_1, \ldots, E_n$ of disjoint sets whose union is E, just set $E_j = \emptyset$ for $j > n$. Then $E_1, E_2, \ldots$ is an infinite sequence of disjoint sets whose union is E and

$$\sum_{j=1}^{n} |\nu(E_j)| = \sum_{j=1}^{\infty} |\nu(E_j)|.$$

Thus, the set of numbers in (A) is a subset of the set of numbers in (B). Since the sup over a larger set is larger, $\mu_1 \le \mu_2$.

To see that $\mu_2 \le \mu_3$, let $E_1, E_2, \ldots$ be a sequence of disjoint sets whose union is E. Now each $\nu(E_j)$ is a complex number, so there is a complex number ω_j such that $|\omega_j| = 1$ and $\omega_j \nu(E_j) = |\nu(E_j)|$. Indeed, we can say

$$\omega_j = \begin{cases} \frac{|\nu(E_j)|}{\nu(E_j)}, & \nu(E_j) \ne 0 \\ 1, & \nu(E_j) = 0. \end{cases}$$

Consider the function f defined by

$$f = \sum_{j=1}^{\infty} \omega_j \chi_{E_j}.$$

If $x \notin E$, $f(x) = 0$. If $x \in E$, it is in exactly one E_j and then $|f(x)| = |\omega_j| = 1$. Thus, $|f| \le 1$ and the partial sums

$$f_n = \sum_{j=1}^{n} \omega_j \chi_{E_j}$$

satisfy $|f_n| \leq 1$ by the same reasoning. We then have

$$\left|\int f_n\,d\nu - \int f\,d\nu\right| = \left|\int (f_n - f)\,d\nu\right| \leq \int |f_n - f|\,d|\nu| \to 0$$

by the dominated convergence theorem, since $f_n \to f$ pointwise and $|f_n - f|$ is bounded above by the constant function 2, which is integrable with respect to the finite measure $|\nu|$. We then have

$$\begin{aligned}\int_E f\,d\nu &= \lim_{n\to\infty} \int f_n\,d\nu \\ &= \sum_{j=1}^{\infty} \int_E \omega_j \chi_{E_j}\,d\nu \\ &= \sum_{j=1}^{\infty} \omega_j \nu(E_j) \\ &= \sum_{j=1}^{\infty} |\nu(E_j)|.\end{aligned}$$

since $\int_E f\,d\nu$ is positive, we have

$$\left|\int_E f\,d\nu\right| = \sum_{j=1}^{\infty} |\nu(E_j)|.$$

Thus, the set of numbers in (B) is a subset of the set in (C), so $\mu_2 \leq \mu_3$.

Now we want to show that $\mu_3 \leq \mu_1$. First, we use Proposition 3.13 (and Exercise 18) to show that μ_3 is finite. If $|f| \leq 1$, then

$$\int |f|\,d|\nu| \leq \int 1\,d|\nu| = |\nu|(X) < \infty,$$

so $f \in L^1(|\nu|) = L^1(\nu)$ and

$$\left|\int_E f\,d\nu\right| \leq \int_E |f|\,d|\nu| \leq \int_E 1\,d|\nu| = |\nu|(E).$$

Thus, $|\nu|(E)$ is an upper bound for the set of numbers in (C), so $\mu_3(E) \leq |\nu|(E) \leq |\nu|(X) < \infty$.

To show that $\mu_3 \leq \mu_1$, let $\varepsilon > 0$ be given. Then we can find some f with $|f| \leq 1$ such that

$$\mu_3(E) - \varepsilon < \left|\int_E f\,d\nu\right|,$$

which implies

$$(*) \qquad \mu_3(E) \leq \left|\int_E f\,d\nu\right| + \varepsilon.$$

We approximate f by a simple function as follows. Let

$$D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

be the closed unit disk in the complex plane. Since $|f| \leq 1$, all the values of f are in D. The collection of balls

$$\{ \mathrm{B}(\varepsilon, z) \mid z \in D \}$$

is an open cover of D. Since D is compact, there is a finite subcover, say

$$\{ \mathrm{B}(\varepsilon, z_j) \mid j = 1, \dots, m \} .$$

Define $B_j = f^{-1}(\mathrm{B}(\varepsilon, z_j)) \subseteq X$, which is measurable since f is measurable. The union of the sets B_j is X. The sets B_j won't be disjoint, but we can use the usual trick and define

$$A_1 = B_1, \qquad A_j = B_j \setminus \bigcup_{k=1}^{j-1} B_k$$

to get a disjoint collection of sets with $A_j \subseteq B_j$ and $\bigcup_j A_j = X$. Define a simple function φ by

$$\varphi = \sum_{j=1}^{m} z_j \chi_{A_j} .$$

This function takes on the values z_j, which are in D, so $|\varphi| \leq 1$. If $x \in X$ it is in exactly one of the sets A_j. But then $f(x) \in \mathrm{B}(\varepsilon, z_j)$, so

$$|f(x) - \varphi(x)| = |f(x) - z_j| < \varepsilon.$$

Thus, $|f - \varphi| < \varepsilon$. We then have

$$\begin{aligned} \left| \left| \int_E f \, d\nu \right| - \left| \int_E \varphi \, d\nu \right| \right| &\leq \left| \left| \int_E f \, d\nu \right| - \left| \int_E \varphi \, d\nu \right| \right| \\ &\leq \left| \int_E f \, d\nu - \int_E \varphi \, d\nu \right| \\ &\leq \int_E |f - \varphi| \, d|\nu| \\ &\leq \int_E \varepsilon \, d|\nu| \\ &= \varepsilon |\nu|(E). \end{aligned}$$

Thus, we have

$$\left| \int_E f \, d\nu \right| \leq \left| \int_E \varphi \, d\nu \right| + \varepsilon |\nu|(E).$$

Substituting this is $(*)$ gives

$$(**) \qquad \mu_3(E) \le \left| \int_E \varphi \, d\nu \right| + \varepsilon + \varepsilon |\nu|(E).$$

Now define $E_j = A_j \cap E$, $j = 1, \ldots, m$. Then the E_j's are a finite sequence of disjoint sets whose union is E and we have

$$\begin{aligned}
\left| \int_E \varphi \, d\nu \right| &= \left| \int_E \left[\sum_{j=1}^m z_j \chi_{A_j} \right] d\nu \right| \\
&= \left| \sum_{j=1}^m z_j \int_E \chi_{A_j} \, d\nu \right| \\
&= \left| \sum_{j=1}^m z_j \int \chi_E \, \chi_{A_j} \, d\nu \right| \\
&= \left| \sum_{j=1}^m z_j \int \chi_{E_j} \, d\nu \right| \\
&= \left| \sum_{j=1}^m z_j \nu(E_j) \right| \\
&\le \sum_{j=1}^m |z_j| |\nu(E_j)| \\
&\le \sum_{j=1}^m |\nu(E_j)| \\
&\le \mu_1(E).
\end{aligned}$$

Substituting this into $(**)$ gives

$$\mu_3(E) \le \mu_1(E) + [1 + |\nu|(E)]\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu_3(E) \le \mu_1(E)$.

We've now shown that $\mu_1 = \mu_2 = \mu_3$. The last step is to show that $\mu_3 = |\nu|$. We've already shown above that $\mu_3(E) \le |\nu|(E)$. To get the reverse inequality, let $g = d\nu/d|\nu|$. We know that $|g| = 1$ $|\nu|$-a.e., and we may as well assume that $|g| = 1$ everywhere (by modifying it on a set of measure zero). Then the conjugate $\bar{g}$ of g satisfies $|\bar{g}| = 1 \le 1$ and so is one of the functions in the definition of μ_3, so

$$\left| \int_E \bar{g} \, d\nu \right| \le \mu_3(E).$$

But, from the definition of g,

$$\begin{aligned}\left|\int_E \bar{g}\, d\nu\right| &= \left|\int_E \bar{g}g\, d|\nu|\right| \\ &= \left|\int_E |g|^2\, d|\nu|\right| \\ &= \left|\int_E 1\, d|\nu|\right| \\ &= |\nu|(E).\end{aligned}$$

Thus, $|\nu|(E) \leq \mu_3(E)$, so $\mu_3 = |\nu|$, and we're done.
