

# HOW TO COMPLETE THE SQUARE

LANCE D. DRAGER

## 1. INTRODUCTION

To complete the square means to take a quadratic function  $f(x) = ax^2 + bx + c$  and do the algebra to write it in the form  $f(x) = a(x - h)^2 + k$ , for some numbers  $h$  and  $k$ . For example,

$$f(x) = -2x^2 + 8x - 7 = -2(x - 2)^2 + 1$$

(check it by working out the square on the right and simplifying).

The advantage of doing this is that it allows you to easily analyze the graph of the quadratic function and its properties.

Here is a brief summary of some of the properties you can read off. Consider the function

$$f(x) = a(x - h)^2 + k$$

- The graph is a parabola. In fact the graph  $y = f(x)$  is the graph  $y = ax^2$  shifted right  $h$  units and up  $k$  units.
- if  $a > 0$  the parabola opens up. If  $a < 0$  the parabola opens down.
- The **vertex** of the parabola is the peak of the mountain if the parabola opens down, or the bottom of the pit if the parabola opens up. The vertex is at the point  $(h, k)$ .
- If the parabola opens down, the maximum value of the function  $f(x)$  is  $k$ , and this maximum occurs when  $x = h$ . The range of the function is  $(-\infty, k]$ . There is no minimum value.  
If the parabola opens up, the minimum value of the function  $f(x)$  is  $k$ , and this minimum occurs when  $x = h$ . The range of the function is  $[k, \infty)$ . There is no maximum value.
- The axis of symmetry of the parabola is the vertical line  $x = h$ .

To see these properties in action, consider Figure 1 and Figure 2.

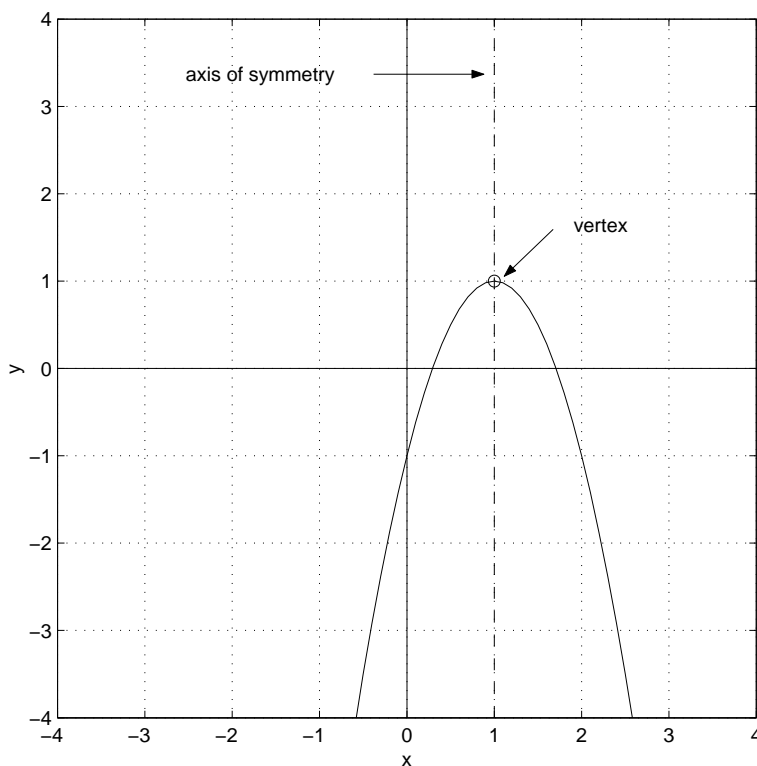
## 2. THE PROCEDURE

We'll now go over a step by step procedure for completing the square, illustrated by the example

$$f(x) = 12x - 3x^2 - 8.$$

**Step 1: Write the function in decreasing powers of  $x$ .** In our example, we write

$$f(x) = -3x^2 + 12x - 8.$$

FIGURE 1. Graph of  $y = -2(x-1)^2 + 1$ 

**Step 2: Factor the coefficient of  $x^2$  out of the first two terms.** In our example, we write

$$(1) \quad f(x) = -3(x^2 - 4x) - 8.$$

**Step 3: Find  $h$  and  $h^2$ .** Inside the parentheses, you'll have two terms that look like  $x^2 + rx$ . Set  $h = -r/2$  and calculate  $h^2$ . Looking at our example in (1) we have  $x^2 + rx = x^2 - 4x$ , so  $r = -4$ . Thus,  $h = -(-4)/2 = 2$  and  $h^2 = 2^2 = 4$ .

**Step 4: Add and subtract  $h^2$  inside the parentheses.** In our example, we look at (1) and write

$$f(x) = -3(x^2 - 4x \underbrace{+4 - 4}) - 8 = -3(x^2 - 4x + 4 - 4) - 8.$$

new terms

**Step 5: Factor the first three terms in the parentheses.** If we've done it right, the first three terms in the parentheses are equal to  $(x-h)^2$ . In our example,  $h = 2$ , so we have

$$f(x) = -3(\underbrace{x^2 - 4x + 4}_{\text{factor}} - 4) - 8 = -3((x-2)^2 - 4) - 8.$$

**Step 6: Multiply the leading coefficient through the parentheses and simplify.** In our example,

$$f(x) = -3((x-2)^2 - 4) - 8 = -3(x-2)^2 - 3(-4) - 8 = -3(x-2)^2 + 12 - 8 = -3(x-2)^2 + 4.$$

So, our final answer is

$$f(x) = -3(x - 2)^2 + 4.$$

**Done!**

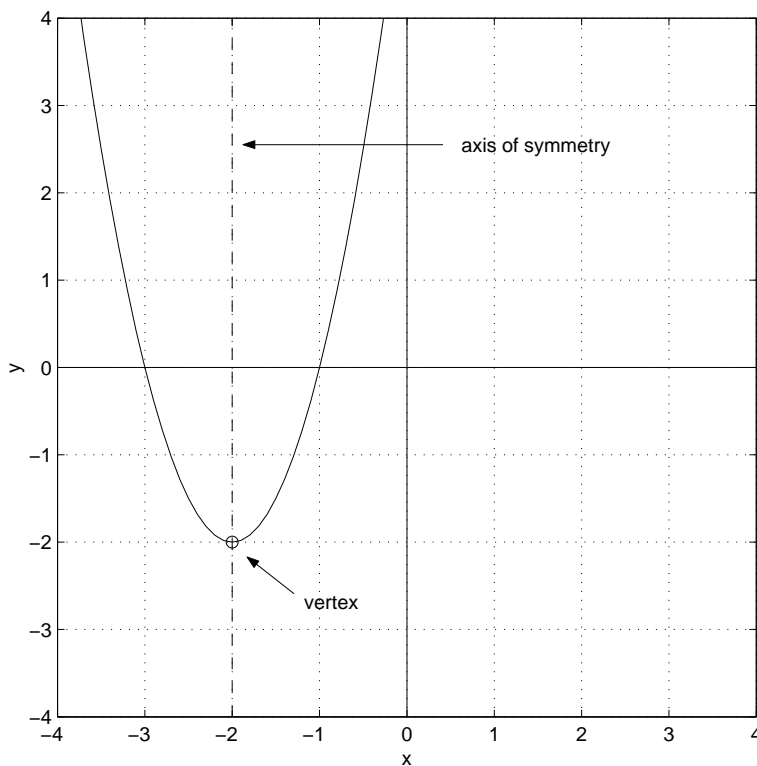


FIGURE 2. Graph of  $y = 2(x + 2)^2 - 2$

### 3. ANOTHER EXAMPLE, STEP BY STEP

. In this example, we wind up with some fractions. Consider the function

$$f(x) = 2x^2 + 6x + 1.$$

**Step 1: Write the function in decreasing powers of  $x$ .** This is already done.

**Step 2: Factor the coefficient of  $x^2$  out of the first two terms.** In the example, we get

$$f(x) = 2(x^2 + 3x) + 1.$$

**Step 3: Find  $h$  and  $h^2$ .** In our example, the coefficient of  $x$  inside the parentheses is  $r = 3$ , so we get  $h = -r/2 = -3/2$  and so  $h^2 = 9/4$ .

**Step 4: Add and subtract  $h^2$  inside the parentheses** This gives us

$$f(x) = 2(x^2 + 3x + 9/4 - 9/4) + 1.$$

**Step 5: Factor the first three terms in the parentheses.** We have  $(x - h) = (x - (-3/2)) = (x + 3/2)$ , and so

$$f(x) = 2(\underbrace{x^2 + 3x + 9/4}_{(x+3/2)^2} - 9/4) + 1 = 2((x + 3/2)^2 - 9/4) + 1.$$

**Step 6: Multiply the leading coefficient through the parentheses and simplify.** In our problem, this gives

$$\begin{aligned} f(x) &= 2((x + 3/2)^2 - 9/4) + 1 \\ &= 2(x + 3/2)^2 - 9/2 + 1 \\ &= 2(x + 3/2)^2 - 9/2 + 2/2 \\ &= 2(x + 3/2)^2 - 7/2, \end{aligned}$$

so

$$f(x) = 2\left(x + \frac{3}{2}\right)^2 - \frac{7}{2}$$

and we are done.

#### 4. AN APPLICATION PROBLEM

Consider the cost and revenue functions from Problem 43 on page 73 of the text. The functions are

$$\begin{aligned} R(x) &= x(50 - 1.25x) \\ C(x) &= 160 + 10x, \end{aligned}$$

both functions have domain  $1 \leq x \leq 40$ . These are the cost and revenue functions for a clock manufacturer. The variable  $x$  represents the production level, in thousands of units,  $C(x)$  is the cost of manufacturing  $x$  thousand units in thousands of dollars, and  $R(x)$  is the revenue (income from sales) that can be expected at a production level of  $x$  thousand units, in thousands of dollars.

**Problem:** find the production level that maximizes profit. Find the maximum profit.

To find the solution, we write down the profit function  $P(x) = R(x) - C(x)$ . Thus, we have

$$P(x) = x(50 - 1.25x) - (160 + 10x) = 50x - 1.25x^2 - 160 - 10x = 40x - 1.25x^2 - 160$$

To find the maximum of this function, we complete the square, following the steps above.

**Step 1: Write the function in decreasing powers of  $x$ .** We get

$$P(x) = -1.25x^2 + 40x - 160.$$

**Step 2: Factor the coefficient of  $x^2$  out of the first two terms.** This gives us

$$P(x) = -1.25(x^2 - (40/1.25)x) - 160 = -1.25(x^2 - 32x) - 160$$

**Step 3: Find  $h$  and  $h^2$ .** In our problem, the coefficient of  $x$  inside the parentheses is  $r = -32$ , so  $h = -r/2 = 16$  and  $h^2 = 16^2 = 256$ .

**Step 4: Add and subtract  $h^2$  inside the parentheses** This gives us

$$P(x) = -1.25(x^2 - 32x + 256 - 256) - 160.$$

**Step 5: Factor the first three terms in the parentheses.** We have  $(x - h) = (x - 16)$ , and so

$$P(x) = -1.25(\underbrace{x^2 - 32x + 256}_{(x-16)^2} - 256) - 160 = -1.25((x - 16)^2 - 256) - 160.$$

**Step 6: Multiply the leading coefficient through the parentheses and simplify.** We have

$$\begin{aligned} P(x) &= -1.25((x - 16)^2 - 256) - 160 \\ &= -1.25(x - 16)^2 - 1.25(-256) - 160 \\ &= -1.25(x - 16)^2 + 320 - 160 \\ &= -1.25(x - 16)^2 + 160. \end{aligned}$$

We've now completed the square to arrive at

$$P(x) = -1.25(x - 16)^2 + 160$$

To complete the problem, we note that the graph of  $P(x)$  is a parabola opening downward with vertex  $(16, 160)$ . Thus, the maximum profit occurs at a production level of  $x = 16$  thousand units. The profit at this production level will be 160 thousand dollars.

As an aide in visualizing what the computations mean, Figure 3 shows the graph of the functions  $R(x)$ ,  $C(x)$  and  $P(x)$ .

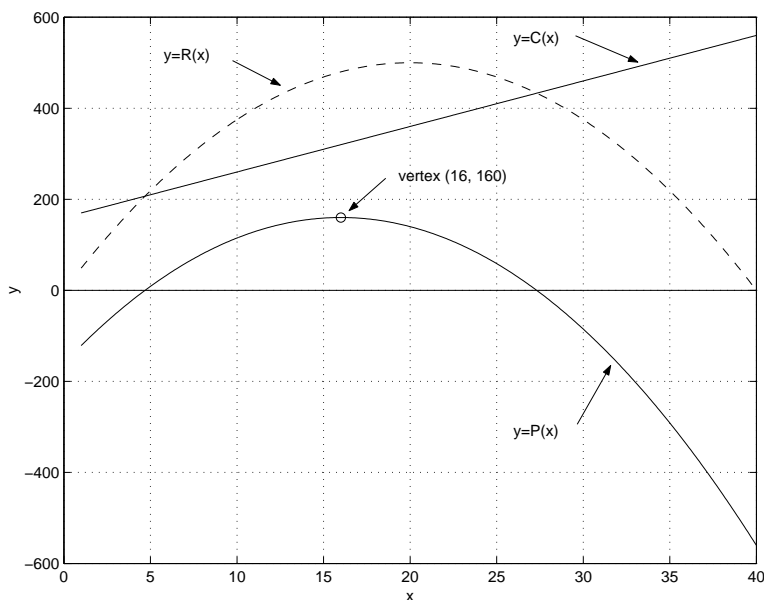


FIGURE 3. The graphs for the application problem.