EXAM

Exam #3

Math 1351-007, Fall 2002

November 22, 2002

ANSWERS

40 pts.

Problem 1. Find the absolute maximum and absolute minimum values of the function $f(x) = x^3 - 6x^2 + 1$ on the interval $[-3, 5]$.

Answer:

To find the critical numbers, we calculate

$$f'(x) = 3x^2 - 12x = 3x(x - 4).$$

Thus, $f'(x) = 0$ at $x = 0$ and $x = 4$. Both of these points are in the interval $[-3, 5]$. Since the interval is closed and bounded, we can find the absolute maximum and minimum by evaluating the function at the endpoints and at the critical points in the interior of the interval. Using a calculator, we get

$$f(-3) - 80$$
$$f(0) = 1$$
$$f(4) = -31$$
$$f(5) = -24.$$

Thus, the absolute maximum of f on $[-3, 5]$ is 1, which occurs at the critical point $x = 0$, and the absolute minimum is -80, which occurs at the endpoint $x = -3$.

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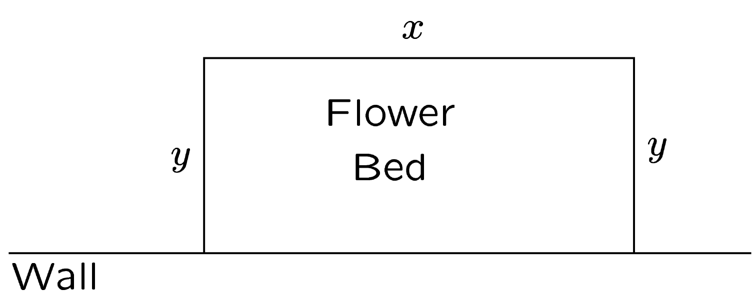
40 pts.

Problem 2. A flower bed is to be constructed. One side of the bed will be formed by the long, straight wall of a building, and the other three sides will be fenced. If there is 8 feet of fencing available, find the dimensions of the flower bed that will give the maximum area.

Be sure to show what interval you are maximizing the function over, and to justify that you have really found the max, not just a critical point.

Answer:

We begin by drawing a picture and labeling the sides of the flower bed.



The area of the flower bed is given by $A = xy$. The length of fence used for the flower bed is $x + 2y$. Since there are 8 feet of fence available, we have the constraint $x + 2y = 8$. Solving the constraint equation for x gives $x = 8 - 2y$ and substituting this in the formula for A gives $A = y(8-2y) = 8y - 2y^2$. Since y is the length of a side of the flower bed, we must have $y \geq 0$. Similarly, we must have $x \geq 0$, so $0 \leq 8 - 2y$, which implies $y \leq 4$.

Thus, our problem is to maximize the function $A = 8y - 2y^2$ over the interval $[0, 4]$. Since this is a closed, bounded interval, we may find the maximum by evaluating the function at the endpoints of the interval and at any critical numbers in the interior of the interval. In our case, $A' = 8 - 4y$, so there is one critical number $y = 2$, which is in our interval $[0, 4]$. Evaluating the function at the relevant points we get

$$A(0) = 0$$
$$A(2) = 8$$
$$A(4) = 0.$$

Thus, the maximum possible area of the flower bed is 8 square feet, which is obtained by making $y = 2$ and $x = 8 - 2(2) = 4$.

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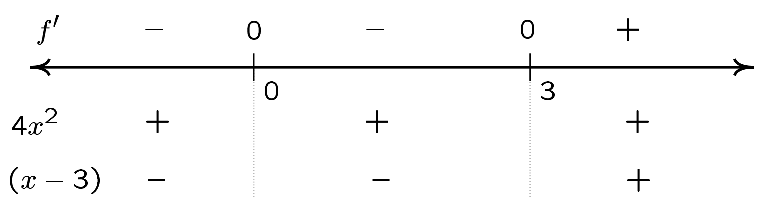
50 pts.

Problem 3. Consider the function $f(x) = x^4 - 4x^3 + 2$.

A. Find all the critical points. Construct the sign table for $f'(x)$. Determine where $f(x)$ is increasing and decreasing. Determine if each critical point is a relative max, a relative min, or neither.

Answer:

We have $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, so the critical numbers are $x = 0, 3$. The sign table is shown below (below the line we show the sign of each factor, used to find the sign of f').



Thus we conclude that f is deceasing on the interval $(-\infty, 3)$ and increasing on the interval $(3, \infty)$. From the first derivative test, the critical point at $x = 0$ is neither a relative max nor a relative min, while the critical point at $x = 3$ is a relative min.

B. Find the sign table for $f''(x)$. Find all inflection points. Determine where the graph $y = f(x)$ is concave up and concave down.

Answer:

We have $f''(x) = 12x^2 - 24x = 12x(x-2)$, which has zeros at $x = 0$ and $x = 2$. If we construct the sign table we get

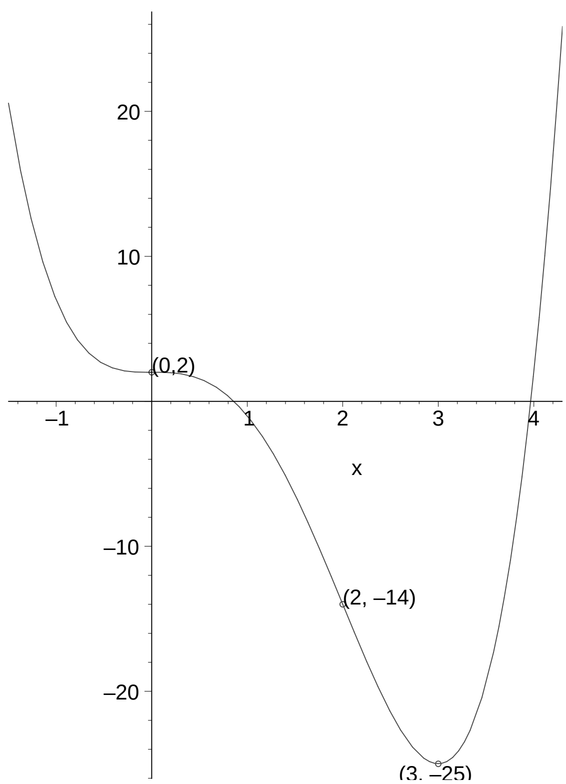
f''	+	0	−	0	+
		0		2	
$12x$	−		+		+
$(x-2)$	−		−		+

Thus, f is concave up on $(-\infty, 0)$ and $(2, \infty)$ and concave down on $(0, 2)$. Both $x = 0$ and $x = 2$ are inflection points.

C. Sketch the graph $y = f(x)$, plotting all critical points and inflection points.

Answer:

Here is a computer sketch of the graph, with the critical points and inflection points marked.



60 pts. **Problem 4**. Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}.$$

To save time, I'll give you the formulas

$$f'(x) = -\frac{2x}{(x^2-1)^2}, \qquad f''(x) = \frac{6x^2+2}{(x^2-1)^3}.$$

A. Find all vertical and horizontal asymptotes for the graph $y = f(x)$.

Answer:

The denominator of $f(x)$ is zero at $x = -1$ and $x = 1$, and the numerator is not zero at these points, so we have vertical asymptotes $x = -1$ and $x = 1$. To find the horizontal asymptotes, we consider the limit of $f(x)$ as $x \to \pm\infty$:

$$\begin{aligned} \lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{x^2}{x^2-1} \\ &= \lim_{x\to\pm\infty} \frac{x^2}{x^2-1} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\pm\infty} \frac{1}{1-\frac{1}{x^2}} \\ &= \frac{1}{1-0} = 1, \end{aligned}$$

so we have a horizontal asymptote $y = 1$.

B. Find all the critical points. Construct the sign table for $f'(x)$. Determine where $f(x)$ is increasing and decreasing. Determine if each critical point is a relative max, a relative min, or neither.

Answer:

From the formula for the first derivative, we see that the only zero of f' is $x = 0$, so that is the only critical number. However, f' is undefined for $x = \pm 1$ (like the function f), so the sign could change at these points. Thus, when making the sign table, we need division points at $= -1, 0, 1$. The sign table is as follows (the symbol $\updownarrow$ indicates a vertical asymptote at that point):

	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
f'	+	↕	+	0	−	↕	−
$(x^2-1)^2$	+		+		+		+
x	−		−		+		+
-2	−		−		−		−

Thus, by the first derivative test, the critical point $x = 0$ is a relative max. The function is increasing on the intervals $(-\infty, -1)$ and $(-1, 0))$ and decreasing on $(0, 1)$ and $(1, \infty)$.

C. Find the sign table for $f''(x)$. Find all inflection points. Determine where the graph $y = f(x)$ is concave up and concave down.

Answer:

From the formula for f'', we see that f'' has no zeros, since the numerator is never zero, so there are no inflection points. However, f'' is undefined at $x = \pm 1$ (like the function f), so we will have to use $x = -1, 1$ as the division points in the sign table. The sign table is as follows:

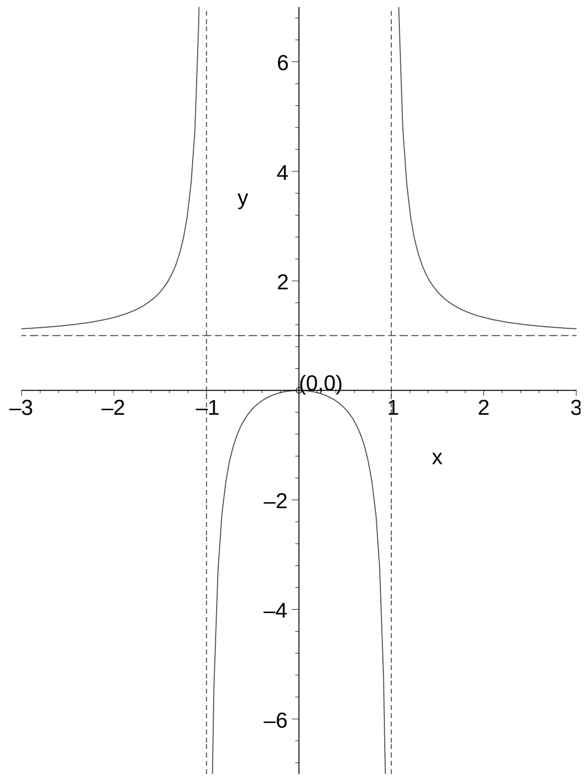
	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
f''	+	↕	−	↕	+
$(x-1)^3$	−		−		+
$(x+1)^3$	−		+		+
$(6x^2+2)$	+		+		+

From this we conclude that the graph $y = f(x)$ is concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$; and concave down on the interval $(-1, 1)$.

D. Sketch the graph $y = f(x)$, plotting all critical points, inflection points, and asymptotes.

Answer:

Here is a computer sketch of the graph. The asymptotes are drawn as dotted lines and the critical point at $(0, 0)$ is marked.



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40 pts.

Problem 5. In each part, find the limit.

A.

$$\lim_{x\to 0} \frac{\sin(3x)}{x+\sin(x)}$$

Answer:

This has indeterminate form 0/0, so we apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin(3x)}{x+\sin(x)} = \lim_{x\to 0} \frac{3\cos(3x)}{1+\cos(x)} = \frac{3(1)}{1+1} = \frac{3}{2}.$$

B.

$$\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$$

Answer:

We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x}{x^3} & & \text{Indeterminate form } \frac{0}{0} \\
&= \lim_{x\to 0} \frac{\cos(x) - 1}{3x} & \text{still } \frac{0}{0} \\
&= \lim_{x\to 0} \frac{-\sin(x)}{6x} & \text{still } \frac{0}{0} \\
&= \lim_{x\to 0} -\frac{\cos(x)}{6} \\
&= -\frac{1}{6}.
\end{aligned}$$

C.

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$$

Answer:

This has indeterminate form ∞/∞. We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} &= \lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} \frac{2x^{1/2}}{2x^{1/2}} \\
&= \lim_{x\to\infty} \frac{2x^{1/2}}{x} \\
&= \lim_{x\to\infty} \frac{2}{x^{1/2}} \\
&= 0
\end{aligned}$$

D.

$$\lim_{x\to 0^+} \sqrt{x}\ln(x)$$

Answer:

This has indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we have to rewrite the expression as a quotient so we get either $0/0$ or ∞/∞. In this case, we can write

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}}},$$

which has the form ∞/∞. The we have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}}} &= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} \\
&= \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \\
&= \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \frac{-2x^{3/2}}{-2x^{3/2}} \\
&= -2 \lim_{x\to 0^+} \frac{x^{3/2}}{x} \\
&= -2 \lim_{x\to 0^+} x^{1/2} \\
&= -2 \lim_{x\to 0^+} \sqrt{x} \\
&= -2(0) = 0.
\end{aligned}$$
